Last time

- Cellular homology
- Compact subspaces of CW complexes

$$\begin{tikzcd} H_n(X_n, X_{n-1}) \arrow[rr, "d_n^{CW}"] \arrow[dr, "\partial"'] & & H_{n-1}(X_{n-1}, X_{n-2}) \\ & H_{n-1}(X_{n-1}) \arrow[ur] & \end{tikzcd}$$

It remains to prove the following.

Lemma If X is a CW complex, then (X_n, X_{n-1}) is a good pair.

For this lemma, we will use a construction of convenient open neighbourhoods.

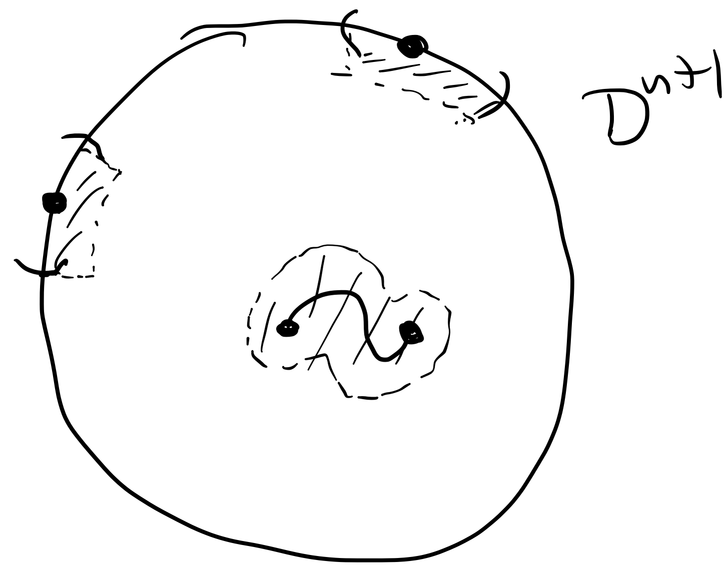
<u>Construction</u> Given $A \subseteq X$ and a function $\varepsilon : \{\text{cells of } X\} \to \mathbb{R}_{>0}$, define $N_\varepsilon(A)$ inductively as follows.

(1) $N^0_\varepsilon(A) = A \cap X_0$.

(2) Given the open neighbourhood $A \cap X_n \subseteq N^n_\varepsilon(A) \subseteq X_n$, define $N^{n+1}_\varepsilon(X)$ by fixing characteristic maps $e_i : D^{n+1} \to X$ and requiring that

$$e_i^{-1}(N_\varepsilon^{n+1}(A)) = \underbrace{e_i^{-1}(N_\varepsilon^{n}(A)) \times (1-\varepsilon(e_i)}_{\text{polar coordinates}}, 1] \cup U(e_i),$$

where $U(e_i)$ is an $\varepsilon(e_i)$-neighborhood of $e_i^{-1}(A) \cap \mathring{D}^{n+1}$.



(3) Set $N_\varepsilon(A) = \bigcup_{n \geq 0} N_\varepsilon^n(A)$.

Since $N_\varepsilon(A) \cap X_n = N_\varepsilon^n(A)$ by construction, $N_\varepsilon(A)$ is open in X.

The result is a direct consequence of the following.

<u>Lemma</u> If $A \subseteq X$ is a subcomplex, then $A \subseteq N_\varepsilon(A)$ is a deformation retract if $\varepsilon < 1$.

<u>Proof</u> We will show that $N_\varepsilon^n(A)$ deformation retracts onto $N_\varepsilon^{n-1}(A) \cup A \cap X_n$ for every $n > 0$. Performing this deformation retraction in the interval $[1/2^n, 1/2^{n-1}]$ gives the result.

For every n-cell $e_i : D^n \to X$ not contained in A, $e_i^{-1}(A) = e_i^{-1}(N_\varepsilon^{n-1}(A)) \times (1-\varepsilon(e_i), 1]$, which deformation retracts onto $e_i^{-1}(N_\varepsilon^{n-1}(A))$ if $\varepsilon(e_i)<1$. Performing this deformation retraction simultaneously in each cell not in A gives the desired result.

$\square$

Essentially the same argument gives the following (exercise).

<u>Prop</u> For any $x \in X$, $N_\varepsilon(x)$ is contractible if $\varepsilon<1$. In particular, CW complexes are locally contractible, hence locally path connected and semilocally simply connected.

We draw one last conclusion.

<u>Prop</u> CW complexes are Hausdorff.

<u>Proof</u> For $x, y \in X$, $N_\varepsilon(x)$ and $N_\varepsilon(y)$ are disjoint for ε sufficiently small (details are an exercise). $\square$

<u>Rmk</u> The same argument shows that CW complexes are <u>normal</u>, which is to say disjoint closed subspaces have disjoint open neighborhoods.

Before turning to computations of cellular homology, we make one immediate application.

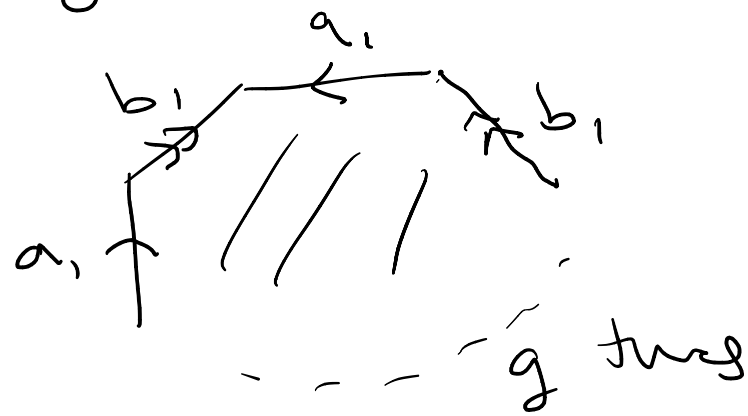
<u>Def</u> Let X be a space such that $H_*(X)$ is finitely generated. The <u>Euler characteristic</u> of X is

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \operatorname{rk} H_i(X).$$

<u>Thm</u> (Euler-Poincaré) If X is a finite CW complex with set of k-cells I_k, then

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i |I_k|.$$

Ex The orientable surface of genus g has $\chi(\Sigma_g) = 2-2g$



Cor In any triangulation of a surface of genus g, we have $V - E + F = 2-2g$

The theorem is a consequence of the following result applied to $C_*^{cw}(X)$.

Lemma If (C, d) is a chain complex of finite rank as an Abelian group, then

$$\sum_{i=0}^{\infty} (-1)^i \operatorname{rk} C_i = \sum_{i=0}^{\infty} (-1)^i \operatorname{rk} H_i(C).$$

Proof The short exact sequences

$$0 \to \ker(d_i) \longrightarrow C_i \xrightarrow{d_i} \operatorname{im}(d_i) \to 0$$

$$0 \to \operatorname{im}(d_{i+1}) \to \ker(d_i) \to H_i(C) \to 0$$

give $\operatorname{rk} C_i = \operatorname{rk} \operatorname{im}(d_i) + \operatorname{rk} \ker(d_i)$ and $\operatorname{rk} H_i(C) = \operatorname{rk} \ker(d_i) - \operatorname{rk} \operatorname{im}(d_{i+1})$. $\square$