Last time

- Open neighbourhoods in CW complexes
- Euler characteristic

Q How to compute d_n^{CW}?

$$\mathbb{Z}\langle n\text{-cells } e^n_\alpha : D^n \to X\rangle \xrightarrow{d_n^{CW}} \mathbb{Z}\langle (n-1)\text{-cells } e^{n-1}_\beta : D^{n-1} \to X\rangle$$

$$e^n_\alpha \longmapsto \sum_\beta c_{\alpha\beta} e^{n-1}_\beta$$

To calculate the coefficients $c_{\alpha\beta}$, we first clarify an issue regarding canonicity of generators for the sphere.

Construction

(0) $H_0(D^0)$ has a canonical generator given by the unique map $\Delta^0 \to D^0$.

(1) We choose the generator for $\tilde{H}_0(S^0)$ corresponding to $(1,-1)$ in the diagram

$$\begin{array}{ccc} H_0(S^0) \cong H_0(\{+1\}) \oplus H_0(\{-1\}) \cong & \mathbb{Z} \oplus \mathbb{Z} \\ \cup\!\!\mid & \cup\!\!\mid \\ \tilde{H}_0(S^0) \xlongequal{\sim} & \{(a,b) : a+b=0\}. \end{array}$$

(2) Given a generator for $\tilde{H}_{n-1}(S^{n-1})$, we obtain a generator for $H_n(D^n, S^{n-1})$ via the isomorphism

$$0 \to H_n(D^n, S^{n-1}) \xrightarrow{\delta} \tilde{H}_{n-1}(S^{n-1}) \to 0$$

(3) Given a generator for $H_n(D^n, S^{n-1})$, we obtain a generator for $\tilde{H}_n(D^n/S^{n-1})$ via the isomorphism

$$H_n(D^n, S^{n-1}) \xrightarrow{q_*} \tilde{H}_n(D^n/S^{n-1}).$$

(4) Given a generator for $\tilde{H}_n(D^n/S^{n-1})$, we obtain a generator for $\tilde{H}_n(S^n)$ via the homeomorphism $D^n/S^{n-1} \cong S^n$ induced by the map

$$\begin{tikzcd}
(v,r) \arrow[rr, mapsto] \arrow[d, mapsto] & & (\sin(r\pi)v, \cos(r\pi)) \\
rv & S^{n-1} \times [0,1] \arrow[r] \arrow[d] & S^n \\
 & D^n \arrow[ur, dashed] &
\end{tikzcd}$$

Returning to the problem at hand, the following diagram commutes by naturality:

$$
\begin{tikzcd}
e^n_\alpha & H_n(X_n, X_{n-1}) \arrow[r, "\delta"] & H_{n-1}(X_{n-1}) \\
& H_n(D^n, S^{n-1}) \arrow[u, "(e^n_\alpha)_*"] \arrow[r, "\delta"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[u, "(e^n_\alpha)_*"'] \\
1 \arrow[uu, mapsto] \arrow[rr, mapsto] & & 1
\end{tikzcd}
$$

So $d_n^{CW}(e^n_\alpha)$ is the image of the canonical generator under the top composite in the diagram

$$
\begin{tikzcd}
1 \arrow[rrr, mapsto] & & & d_n^{CW}(e^n_\alpha) \arrow[ddd, mapsto] \\
\tilde{H}_{n-1}(S^{n-1}) \arrow[r, "(e^n_\alpha)_*"] \arrow[dddrr] & \tilde{H}_{n-1}(X_{n-1}) \arrow[r] & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[d, "\cong"] & \\
1 \arrow[dddrr, mapsto] & & \tilde{H}_{n-1}(X_{n-1}/X_{n-2}) \arrow[d, "\cong"] & \\
& & \tilde{H}_{n-1}\left(\bigvee_\beta S^{n-1}\right) \arrow[d, "(\pi_\beta)_*"] & \sum_\beta c_{\alpha\beta} e_\beta^{n-1} \arrow[d, mapsto] \\
& & \tilde{H}_{n-1}(S^{n-1}) & c_{\alpha\beta} \\
& & c_{\alpha\beta} &
\end{tikzcd}
$$

Prop (cellular boundary formula) Let $f_{\alpha\beta}$ be the composite

$$S^{n-1} \xrightarrow{e_\alpha^n|_{S^{n-1}}} X_{n-1} \twoheadrightarrow X_{n-1} \Big/ X_{n-1} \setminus e_{n-1}^{\beta}(\mathring{D}^{n-1}) \cong S^{n-1}.$$

For $n>1$, $d_n^{CW}(e_\alpha^n) = \sum_\beta \deg(f_{\alpha\beta})\, e_\beta^{n-1}$.

Rmk $d_1^{CW}(e_\alpha^1)$ is the difference of the endpoints of e_α^1.

This formula is only as useful as our ability to compute degrees.

Observation For any $x \in S^m$ and open $x \in U$, the homomorphisms

$$H_m(S^m) \to H_m(S^m, S^m \setminus \{x\}) \leftarrow H_m(U, U \setminus \{x\})$$

are isomorphisms.

Def Given $f : S^m \to S^m$ and $y \in S^m$ such that $f^{-1}(y)$ is finite, the <u>local degree</u> of f at $x \in f^{-1}(y)$, denoted $\deg_x(f)$, is the image of 1 under the composite

$$\mathbb{Z} \cong H_m(U_x, U_x \setminus \{x\}) \to H_m(V, V \setminus \{y\}) \cong \mathbb{Z},$$

where $x \in U_x$ and $y \in V$ are open neighbourhoods such that $f(x') \neq y$ for $x \neq x' \in U_x$.

Exercise $\deg_x(f)$ is independent of choices.

Prop Let $f: S^m \to S^m$ be any map. If $|f^{-1}(y)| < \infty$, then

$$\deg(f) = \sum_{x \in f^{-1}(y)} \deg_x(f).$$

Proof Changing the neighborhoods U_x to be pairwise disjoint, we have the commuting diagram

$$\begin{tikzcd}
H_m(S^m) \arrow[r, "f_*"] \arrow[d] & H_m(S^m) \arrow[d, "\cong"] \\
H_m(S^m, S^m \setminus f^{-1}(y)) \arrow[r] & H_m(S^m, S^m \setminus \{y\}) \\
\bigoplus_{x \in f^{-1}(y)} H_m(U_x, U_x \setminus \{x\}) \arrow[r] \arrow[u, "\cong"] & H_m(V, V \setminus \{y\}) \arrow[u, "\cong"]
\end{tikzcd}$$

$\square$