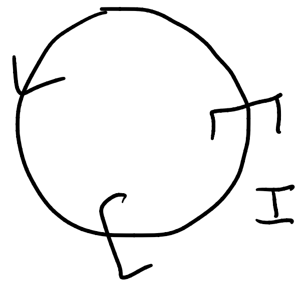
__Last time__ $S^{n-1} \xrightarrow{e^n_\alpha} X_{n-1} \to X_{n-1}/X_{n-1} \setminus e^{n-1}_\beta(\mathring{D}^{n-1}) \cong S^{n-1}$

- Cellular boundary formula
- Local degrees

__Ex__ Consider the map $S^1 \xrightarrow{f} S^1$ given by collapsing an interval of arc:

$f:$ 

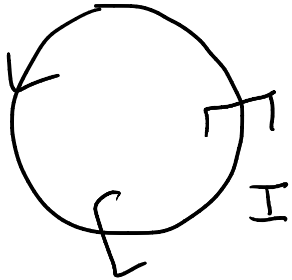
$\longrightarrow S^1/_I \cong D^1/_{S^0} \cong S^1$

For $x \notin I$, $\deg_x(f) = 1$ in view of the diagram

$$\begin{tikzcd}
(S^1 \setminus I,\ S^1 \setminus I \cup \{x\}) \arrow[r, "f"] & (S^1 \setminus \{y_0\},\ S^1 \setminus \{f(x), y_0\}) \\
\big((\theta_1, \theta_2),\ (\theta_1, \theta) \cup (\theta, \theta_2)\big) \arrow[u, "\cong"'] \arrow[r, "\text{scale}"] & \big((0, 2\pi),\ (0, \varphi) \cup (\varphi, 2\pi)\big) \arrow[u, "\cong"'] \\
\big((0,1),\ (0, 1/2) \cup (1/2, 1)\big) \arrow[u, "\text{piecewise linear}", "\cong"'] \arrow[r, equal] & \big((0,1),\ (0, 1/2) \cup (1/2, 1)\big) \arrow[u, "\text{piecewise linear}", "\cong"']
\end{tikzcd}$$

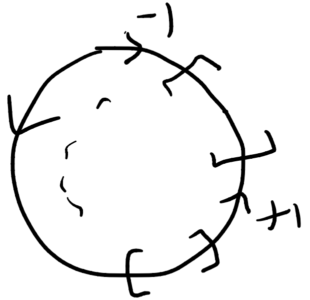
Hence $\deg(f) = 1$.

<u>Ex</u> Consider the map $S^1 \xrightarrow{f'} S^1$ given by collapsing an interval of arc and reversing orientation:

f':  $\longrightarrow S^1/I \cong D^1/S^0 \xrightarrow{-1} D^1/S^0 \cong S^1$

This map differs from f by a reflection, so $\deg(f') = -1$.

<u>Ex</u> Consider the map $S^1 \xrightarrow{f''} S^1$ obtained by collapsing k intervals of arc and folding, with orientations determined by the tuple $(\varepsilon_1, \dots, \varepsilon_k) \in \{-1,1\}^k$:

f'': 

$$\begin{tikzcd} \arrow[r] & S^1/\bigcup_{i=1}^{k} I_i \arrow[r, "\cong"] & \bigvee_k D^1/S^0 \arrow[r, "\vee \varepsilon_i"] & \bigvee_k D^1/S^0 \arrow[r, "\cong"] & \bigvee_k S^1 \arrow[d, "\text{fold}"] \\ & & & & S^1 \end{tikzcd}$$

By local degrees and the previous examples, we have $\deg(f'') = \sum_{i=1}^{k} \varepsilon_i$.

<u>Prop</u> For $g>0$,

$$H_n(\Sigma_g) \cong \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}^{2g} & n=1 \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u> We calculate cellular homology:

$$\mathbb{Z}\langle P\rangle \xrightarrow{d} \mathbb{Z}\langle a_i, b_i \mid 1\le i\le g\rangle \xrightarrow{0} \mathbb{Z}\langle v\rangle$$

The coefficient of a_1 in dP is the degree of the map obtained by collapsing $\left[\frac{2\pi}{4g}, \frac{4\pi}{4g}\right]$ and $\left[\frac{6\pi}{4g}, 2\pi\right]$ with orientations $(\varepsilon_1, \varepsilon_2) = (1,-1)$, which is $1-1=0$ by the example. By symmetry $dP=0$, implying the claim. $\square$

Essentially the same argument establishes a much more general result.

<u>Prop</u> Let X be a 2-dimensional CW complex obtained by attaching polygons P_i to a wedge of circles a_j. The coefficient of a_j in dP_i is the sum of the exponents of a_j in the boundary word of P_i.

This result permits the construction of a non-contractible, acyclic space.

<u>Example</u> Let X be obtained from $S^1 \vee S^1$ by attaching two 2-cells using the words $a^5 b^{-3}$ and $b^3 (ab)^{-2}$. Then $d_2 : \mathbb{Z}^2 \to \mathbb{Z}^2$ has matrix $\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$, which has determinant -1, so d_2 is an isomorphism (or exercise), so $H_2(X) = \ker d_2 = 0$ and $H_1(X) = \ker d_1 / \operatorname{im} d_2 = 0$

$$\Longrightarrow \tilde{H}_*(X) = 0.$$

(But X is not contractible, since $\pi_1(X) = \langle a, b \mid a^5 b^{-3}, b^3 (ab)^{-2} \rangle$, which surjects onto the symmetries of a dodecahedron (Hatcher))

Def (Real) <u>projective n-space</u> $\mathbb{RP}^n$ is the set of lines through 0 in $\mathbb{R}^{n+1}$, topologized with the quotient topology from the function

$$\begin{aligned} S^n &\xrightarrow{q} \mathbb{RP}^n \\ \vec{v} &\longmapsto \{t\vec{v} : t \in \mathbb{R}\}. \end{aligned}$$

We call $\mathbb{RP}^1$ and $\mathbb{RP}^2$ the real <u>projective line</u> and <u>projective plane</u>, respectively.

Note that $q^{-1}(\{t\vec{v} : t \in \mathbb{R}\}) = \left\{ \frac{\vec{v}}{\|\vec{v}\|}, \frac{-\vec{v}}{\|\vec{v}\|} \right\}$.

Ex $\mathbb{RP}^0 = \{\mathbb{R}\}$, a singleton.

Ex $\mathbb{RP}^1$ is homeomorphic to S^1 via the map

$$\begin{tikzcd} z \arrow[r, mapsto] & z^2 \\ \mathbb{C} \supseteq S^1 \arrow[r] \arrow[d, "q"'] & S^1 \\ \mathbb{RP}^1 \arrow[ur, dashed] & \end{tikzcd}$$

Which is continuous (by properties of the quotient topology) and bijective (exercise) with compact source and Hausdorff target.

Prop For $m>0$,

$$H_n(\mathbb{RP}^m) \cong \begin{cases} \mathbb{Z} & n=0 \text{ or } n=m \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0<n<m \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$