Last time

- $H_*(\Sigma_g)$
- Homology of 2-cell complexes
- Real projective space

$\mathbb{RP}^n = S^n/_{x \sim -x}$

Prop For $m > 0$,

$$H_n(\mathbb{RP}^m) \cong \begin{cases} \mathbb{Z} & n = 0 \text{ or } n = m \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0 < n < m \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Proof We have the standard inclusions

$$
\begin{tikzcd}
\mathbb{R}^m \arrow[r, hook, "x \mapsto (x,0)"] & \mathbb{R}^{m+1} \\
S^{m-1} \arrow[u, phantom, "\cup\!|"] \arrow[r, hook, "\text{equator}"] \arrow[d, "\pi_{m-1}"'] & S^m \arrow[u, phantom, "\cup\!|"] \arrow[d, "\pi_m"] \\
\mathbb{R}P^{m-1} \arrow[r, dashed] & \mathbb{R}P^m ,
\end{tikzcd}
$$

and
$$
\begin{aligned}
\mathbb{R}P^m \setminus \mathbb{R}P^{m-1} &= (S^m \setminus S^{m-1})/_{x\sim -x} \\
&= \mathring{D}^m_+ \cup \mathring{D}^m_- /_{x \sim -x} \\
&\cong \mathring{D}^m_+
\end{aligned}
$$

It follows (exercise) that the diagram

$$\begin{tikzcd}
S^{m-1} \arrow[r, hook] \arrow[dd, two heads, "q_{m-1}"'] & D^m_+ \arrow[d, phantom, "\cap"] \arrow[dd, bend left=60, "e^m"] \\
 & S^m \arrow[d, "q_m"] \\
\mathbb{RP}^{m-1} \arrow[r, hook] & \mathbb{RP}^m
\end{tikzcd}$$

expresses $\mathbb{RP}^m$ as obtained by attaching a single m-cell to $\mathbb{RP}^{m-1}$ along the quotient map q_{m-1}. By induction, then, $\mathbb{RP}^m$ is a CW complex with a single n-cell for each $0 \le n \le m \le \infty$, resulting in the cellular chain complex

$$0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0,$$

where $d_n(1)$ is the degree of the composite $f: S^{n-1} \to S^{n-1}$ obtained from the diagram

$$\begin{tikzcd}
D^n \arrow[r, "e^n"] & \mathbb{RP}^n & & \\
S^{n-1} \arrow[u, hook] \arrow[r, "q_{n-1}"] & \mathbb{RP}^{n-1} \arrow[u, hook] \arrow[r, two heads] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} & \\
& D^{n-1}_+ \arrow[u, "e^{n-1}"] \arrow[u, bend right=50, "q_{n-1}"', "\subseteq S^{n-1}"] \arrow[r] & D^{n-1}_+/S^{n-2} \arrow[u, dashed, "\cong"'] \arrow[r, equal, "\text{canonical}", "\sim"'] & S^{n-1}
\end{tikzcd}$$

We have $f^{-1}(P_N) = \{P_N, P_S\}$ (the poles), and the restrictions of f to $\mathring{D}^n_-$ and $\mathring{D}^n_+$ are homeomorphisms differing by the antipodal map. Hence

$$\begin{aligned}
\deg(f) &= \deg_{P_N}(f) + \deg_{P_S}(f) \\
&= \deg_{P_N}(f) + (-1)^n \deg_{P_N}(f) \\
&= \pm\left[1+(-1)^n\right] \\
&= \begin{cases} \pm 2 & n \text{ even} \\ 0 & n \text{ odd}. \end{cases}
\end{aligned}$$

Calculating the homology of the complex

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0,$$

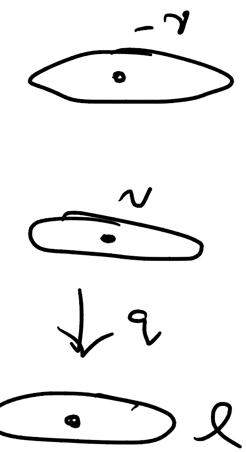
we find that

$$H_n(\mathbb{RP}^m) \cong \begin{cases} \mathbb{Z} & n=0 \text{ or } n=m \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0<n<m \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

<u>Rmk/Spoiler</u> This example shows that some signs matter and some do not. A theory without signs is available in homology with <u>coefficients</u> in $\mathbb{Z}/2\mathbb{Z}$. To be continued...

Before moving on, we extract a definition from this example.

Given a line $\ell \in \mathbb{RP}^n$ and $v \in \ell$, $B_\varepsilon(v) \cap S^{n-1}$ contains no pair of antipodal points for small ε (e.g., $\varepsilon < 1$), so $q|_{B_\varepsilon(v) \cap S^{n-1}}$ is injective, and similarly for $-v$. Thus, $q|_{B_\varepsilon(\pm v) \cap S^{n-1}}$ is a homeomorphism onto its image, since q is an open map; indeed, for $U \subseteq S^n$ open, $q^{-1}(q(U)) = U \cup -U$.



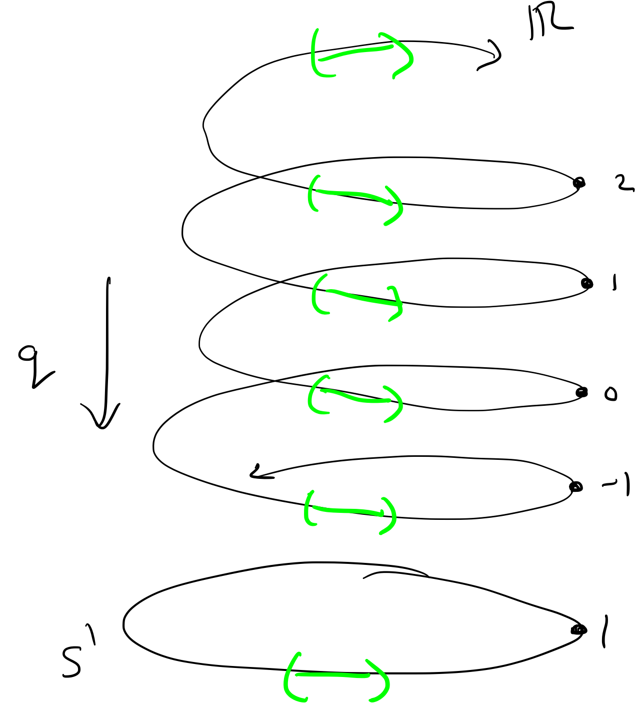
Def Let $p: E \to B$ be a surjective map.

(1) We say that an open $U \subseteq B$ is <u>evenly covered</u> by p if $p^{-1}(U) = \bigcup_{i \in I} V_i$ with

(a) $V_i \subseteq E$ open

(b) $V_i \cap V_j = \phi$ for $i \neq j$

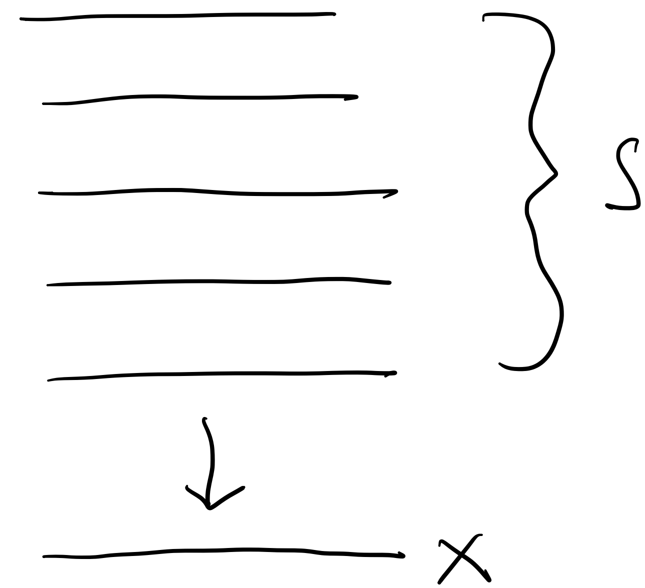
(c) $p|_{V_i} : V_i \to p(V_i)$ a homeomorphism.



(2) We say that p is a <u>covering map</u>, and E a <u>covering space</u> of B, if every $b \in B$ has an open neighborhood evenly covered by p.

Ex $q_n : S^n \to \mathbb{RP}^n$ is a covering map.

Ex For any X, id_X is a covering map. More generally, $X \times S \xrightarrow{p_X} X$ is a covering map for any set S.



Here is a more interesting example.

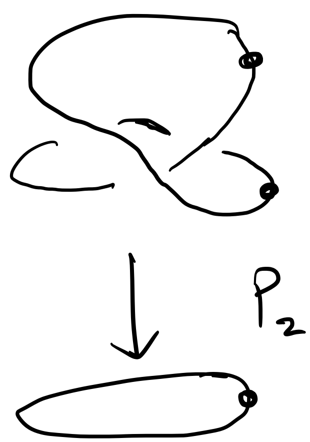
Ex $q: \mathbb{R} \to S^1$ is a covering map.
Indeed, if $U = \{e^{2\pi i x} : x \in (0, 1/2)\}$,

$$q^{-1}(U) = \bigcup_{n \in \mathbb{Z}} (n, n+1/2),$$

and we claim that q induces a homeomorphism $(n, n+1/2) \cong U$. Since $\cos(2\pi x)$ is monotonically decreasing on $[n, n+1/2]$, so $q|_{[n,n+1/2]}$ injects (and surjects) onto $\overline{U}$. Since $[n, n+1/2]$ is compact and $U \subseteq S^1 \subseteq \mathbb{R}^2$ is Hausdorff (exercise), $[n, n+1/2] \cong \overline{U}$, so $(n, n+1/2) \cong U$.

The same argument applies to $x \in (1/2, 1)$, $x \in (1/4, 3/4)$, and $x \in (-1/4, 1/4)$.

<u>Ex</u> The map $p_n : S^1 \to S^1$ given by $p(z) = z^n$ is a covering map for any $n \neq 0$.



Ex $q|_{\mathbb{R}_{>0}}$ is not a covering map (exercise).

Construction Let $p: E \to B$ be a covering map, $U \subseteq B$ evenly covered by p, and $p|_V$ a homeomorphism onto V. Given $\sigma: \Delta^n \to B$ with $\operatorname{im}(\sigma) \subseteq U$, we write

$$\tilde{\sigma}_V = (p|_V)^{-1} \circ \sigma : \Delta^n \longrightarrow V \subseteq E.$$

We write $\tilde{\sigma} = \sum_i \tilde{\sigma}_{V_i}$, where $p^{-1}(U) = \bigcup_i V_i$, well-defined if p is a finite cover.

Rmk $\tilde{\sigma}$ is independent of $U \supseteq \operatorname{im}(\sigma)$.

Def Let $p: E \to B$ be a finite covering map. The transfer associated to p is the homomorphism $\tau: H_*(B) \to H_*(E)$ induced by the chain map (exercise)

$$\begin{array}{ccccc} C_*(X) & \xrightarrow{\text{subdivide}} & C_*^{\mathcal{O}}(X) & \longrightarrow & C_*(Y), \\ & & \sigma & \longmapsto & \tilde{\sigma} \end{array}$$

where $\mathcal{O}$ is any cover of X by evenly covered neighbourhoods.

Exercise τ is independent of $\mathcal{O}$.

Exercise Show that, if p is a degree k cover, then $p_* \circ \tau$ is multiplication by k.