Last time

- $H_*(\mathbb{RP}^n)$
- Covering spaces
- Transfer $\tau : H_*(B) \to H_*(E)$

Recall that τ is induced by the chain map

$$C_*(B) \xrightarrow{\text{subdivide}} C_*^{\theta}(B) \longrightarrow C_*(E),$$
$$\sigma \longmapsto \tilde{\sigma}$$

where θ is any open cover by evenly covered neighborhoods. This map is natural:

<u>Lemma</u> Consider the diagrams

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{f}} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array} \qquad \begin{array}{ccc} H_*(E_1) & \xrightarrow{\tilde{f}_*} & H_*(E_2) \\ {\scriptstyle \tau_1}\uparrow & & \uparrow{\scriptstyle \tau_2} \\ H_*(B_1) & \xrightarrow{f_*} & H_*(B_2) \end{array}$$

where p_1 and p_2 are finite covers. If the first diagram commutes and $\tilde{f}$ is bijective on fibers, then the second diagram commutes.

<u>Proof</u> Fix $\mathcal{O}_2$ for B_2, and choose $\mathcal{O}_1$ for B_1 such that $f(U)$ lies in some element of $\mathcal{O}_2$ for each $U \in \mathcal{O}_1$.

Then $f_\#(C_*^{\mathcal{O}_1}(B)) \subseteq C_*^{\mathcal{O}_2}(B)$, and the bottom square of the diagram

$$\begin{tikzcd}
C_*(B_1) \arrow[r, "f_\#"] \arrow[d, "\text{subd.}"] & C_*(B_2) \arrow[d, "\text{subd.}"] \\
C_*^{\mathcal{O}_1}(B_1) \arrow[r, "f_\#"] \arrow[d] & C_*^{\mathcal{O}_2}(B_2) \arrow[d] \\
C_*(E_1) \arrow[r, "\bar{f}_*"] & C_*(E_2)
\end{tikzcd}$$

commutes by our assumption on $\tilde{f}$. Since the top square commutes up to chain homotopy, the claim follows. $\square$

Idea: Avoid signs by working over $\mathbb{Z}/2\mathbb{Z}$.

Def: The group of singular n-chains with coefficients in G, an Abelian group, is

$$C_n(X;G) := C_n(X) \otimes G \cong \bigoplus_{\sigma:\Delta^n \to X} G.$$

With the same formula for ∂, $(C_*(X;G), \partial)$ is a chain complex, whose homology is called homology with coefficients in G. Everything true about $H_*(X)$ is true about $H_*(X;G)$, with the same proof:

(1) Homotopy invariance

(2) Subdivision

(3) LES of a pair

(4) Mayer-Vietoris

(5) Excision

(6) Cellular homology

(7) Transfer

Computations look a little different.

<u>Ex</u>

$$\tilde{H}_n(S^m; G) \cong \begin{cases} G & m=n \\ 0 & \text{otherwise.} \end{cases}$$

Degree theory works about the same.

<u>Exercise</u> For any map $f: S^n \to S^n$, the homomorphism $f_*: H_n(S^n; G) \to H_n(S^n; G)$ is multiplication by $\deg(f)$.

Thus, the coefficients in the cellular boundary formula are independent of G.

<u>Prop</u> For $m \geq 0$,

$$H_n(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & 0 \leq n \leq m \\ 0 & \text{otherwise} \end{cases}$$

Proof The cellular chain complex is

$$\cdots \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \to 0.$$

$\square$

Using mod 2 coefficients, the transfer homomorphism fits into an exact sequence.

Prop Let $p: E \to B$ be a degree 2 covering map. There is a canonical long exact sequence of the form

$$\cdots \to H_n(B;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau} H_n(E;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_*} H_n(B;\mathbb{Z}/2\mathbb{Z}) \to H_{n-1}(B;\mathbb{Z}/2\mathbb{Z}) \to \cdots$$

Proof Choose Θ for B, and let $\tilde{\Theta}$ denote the collection of $V \subseteq E$ such that $p|_V$ is a homeomorphism onto an element of Θ. The sequence

$$0 \to C_*^{\Theta}(B;\mathbb{Z}/2\mathbb{Z}) \to C_*^{\tilde{\Theta}}(E;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_\#} C_*^{\Theta}(B;\mathbb{Z}/2\mathbb{Z}) \to 0$$
$$\sigma \longmapsto \tilde{\sigma}$$

is exact. $\square$

Thm (Borsuk) An odd map $S^n \to S^m$ has odd degree.

Proof Supposing that $\tilde{f}(-x) = -\hat{f}(x)$ for $x \in S^n$;

we obtain the induced map

$$\begin{tikzcd} S^n \arrow[r, "\tilde{f}"] \arrow[d, "q"'] & S^m \arrow[d, "q"] \\ \mathbb{RP}^n \arrow[r, dashed, "f"] & \mathbb{RP}^m \end{tikzcd}$$

whence the commutative diagram with exact rows (coefficients all $\mathbb{Z}/2\mathbb{Z}$)

$$\begin{tikzcd} H_n(\mathbb{RP}^m) \arrow[r, "\tau"] \arrow[d, "f_*"] & H_n(S^m) \arrow[r, "p_*"] \arrow[d, "\tilde{f}_*"] & H_n(\mathbb{RP}^m) \arrow[r, "\delta"] \arrow[d, "f_*"] & H_{n-1}(\mathbb{RP}^m) \arrow[r, "\tau"] \arrow[d, "f_*"] & H_{n-1}(S^m) \arrow[d, "\tilde{f}_*"] \\ H_n(\mathbb{RP}^m) \arrow[r, "\tau"] & H_n(S^m) \arrow[r, "p_*"] & H_n(\mathbb{RP}^m) \arrow[r, "\delta"] & H_{n-1}(\mathbb{RP}^m) \arrow[r, "\tau"] & H_{n-1}(S^m) \end{tikzcd}$$

By exactness, τ is injective in degree m, hence an isomorphism. Since p_* is an isomorphism in degree 0, $\tau = 0$ in this degree, and it follows by exactness that δ is an isomorphism for $0 < n \leq m$. Since f_* and $\tilde{f}_*$ are isomorphisms in degree 0, it follows by induction that f_* is an isomorphism in all degrees, so $\tilde{f}_*$ is an isomorphism in degree m. But, in this degree, $\tilde{f}_*$ is multiplication by $\deg(\tilde{f})$. $\square$

<u>Cor</u> (Borsuk-Ulam theorem) Given $g: S^n \to \mathbb{R}^n$, $g(x) = g(-x)$ for some $x \in S^n$.

Proof Consider the function $g(x) - g(-x)$, If this function has no zero, the composite

$$S^{n-1} \subseteq D^n_+ \subseteq S^n \longrightarrow S^{n-1}$$
$$x \longmapsto \frac{g(x)-g(-x)}{\|g(x)-g(-x)\|}$$

is odd and has degree 0. $\square$

Corollary (Lusternik-Schnirelmann theorem) If $S^n = \bigcup_{i=0}^{n} A_i$ with each A_i open or closed, then $A_i \cap -A_i \neq \emptyset$ for some $0 \leq i \leq n$.

Proof Applying Borsuk-Ulam to the map

$$\begin{aligned} S^n &\xrightarrow{g} \mathbb{R}^n \\ x &\longmapsto \left(\inf_{y \in A_i} |x-y|\right)_{i=1}^{n}, \end{aligned}$$

we obtain $x \in S^n$ such that $g(x) = g(-x)$. WLOG, $x \notin A_{n+1}$, so $x \in A_i$ for some $i \neq n+1$. Thus, $g_i(x) = 0$, so $g(-x) = 0$, and $-x$ is a limit point of A_i. Hence WLOG A_i is open, so $x \in \mathring{A}_i$ and $-x \in \overline{A_i}$, whence $B_\varepsilon(x) \subseteq A_i$ and $B_\varepsilon(x) \cap A_i \neq \phi$ for some $\varepsilon > 0$. Choosing $y \in B_\varepsilon(x) \cap A_i$, we have $\{y, -y\} \subseteq A_i$. $\square$