Last time

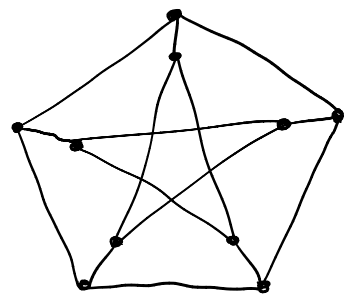
- Homology with coefficients
- Borsuk, Borsuk-Ulam theorems

Cor (Lusternik-Schnirelmann) If $S^n = \bigcup_{i=0}^{n} A_i$ with each A_i open or closed, then $A_i \cap -A_i \neq \phi$ for some $0 \le i \le n$.

We sketch an application of these ideas to combinatorics.

Def A <u>k-coloring</u> of a graph Γ is a function $c: V(\Gamma) \to \{1, \dots, k\}$ such that $c(v) \neq c(w)$ for adjacent v & w. The <u>chromatic number</u> of Γ is the least k for which a k-coloring exists.

Def The <u>Kneser graph</u> $K(n,k)$ has vertices $\{S \subseteq \{1, \dots, n\} \mid |S| = k\}$, with S adjacent to T iff $S \cap T = \emptyset$.



$K(5,2)$

Thm (Lovasz, "Kneser conjecture") The chromatic number of $K(n,k)$ is $n - 2k + 2$ for $k > 0$ and $n \geq 2k - 1$.

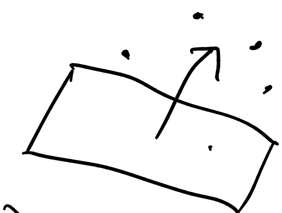
Proof To see that the chromatic number is at most $n-2k+2$, we show that the function

$$c(S) = \min(\min(S), n-2k+2)$$

is a coloring. Indeed, if $c(S) = c(T)$, then either $\min(S) = \min(T)$, so $S \cap T \neq \emptyset$; or $S \cup T \subseteq \{n-2k+2, \ldots, n\}$, a set of cardinality $2k-1$, so $S \cap T \neq \emptyset$.

For the lower bound, set $d = n-2k+1$ and choose points $x_i \in S^d$ for $1 \leq i \leq n$ in general position, so that no hyperplane through $0 \in \mathbb{R}^{d+1}$ contains $d+1$ of the x_i. Fix a coloring c of $K(n,k)$ with d colors, and identify the vertices of $K(n,k)$ with the corresponding subsets of $\{x_1, \dots, x_n\}$. Define $A_j \subseteq S^d$ for $1 \le j \le d$ by declaring that $x \in A_j$ if the hemisphere $\{y \in S^d : \langle x, y \rangle > 0\}$ contains a k-element subset $S \subseteq \{x_1, \dots, x_n\}$ with $c(S) = j$. Define

$$A_{d+1} = S^d \setminus \bigcup_{j=1}^{d} A_j .$$



Then A_j is open for $1 \le j \le d$ and A_{d+1}

is closed. By L-S, $\{x, -x\} \subseteq A_j$ for some $x \in S^d$ and $1 \le j \le d+1$.

If $j \ne d+1$, then the hyperplane perpendicular to x separates sets S and T with $c(S) = c(T)$, so c is not a coloring. If $j = d+1$, then the hemisphere $\{\langle x, y\rangle > 0\}$ contains at most $k-1$ of the x_i, and similarly for $-x$, so the hyperplane perpendicular to x contains at least $n - 2(k-1) = d+1$ points, contrary to our assumption. $\square$

Q Can we understand the homology of a product in terms of the homology of the factors?

At the level of cellular chains, we have

$$
\begin{aligned}
C_n^{CW}(X \times Y) &\cong \mathbb{Z}\langle n\text{-cells of } X \times Y \rangle \\
&\cong \mathbb{Z}\Big\langle \coprod_{p+q=n} \left\{ \begin{matrix} p\text{-cells} \\ \text{of } X \end{matrix} \right\} \times \left\{ \begin{matrix} q\text{-cells} \\ \text{of } Y \end{matrix} \right\} \Big\rangle \\
&\cong \bigoplus_{p+q=n} \mathbb{Z}\Big\langle \begin{matrix} p\text{-cells} \\ \text{of } X \end{matrix} \Big\rangle \otimes \mathbb{Z}\Big\langle \begin{matrix} q\text{-cells} \\ \text{of } Y \end{matrix} \Big\rangle \\
&\cong \bigoplus_{p+q=n} C_p^{CW}(X) \otimes C_q^{CW}(Y).
\end{aligned}
$$

Def The *tensor product* of the chain complexes C and D is the chain complex $C \otimes D$ with $(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$ and

$$\partial(c \otimes d) = \partial c \otimes d + (-1)^{|c|} c \otimes \partial d.$$

Dream 1 $C_*(X \times Y)$ is quasi-isomorphic to $C_*(X) \otimes C_*(Y)$.

Although there is no obvious map in either direction, this dream is true.

Dream 2 $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.

This dream is only sometimes. In light of Dream 1, however, it is pretty algebraic.

Thm (Künneth) Let C and D be chain complexes of free Abelian groups. If $H_*(C)$ is also free, then

$$H_n(C \otimes D) \cong \bigoplus_{p+q=n} H_p(C) \otimes H_q(D).$$

Ex $C = (\mathbb{Z} \xrightarrow{2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z})$

$D = \mathbb{Z}/2\mathbb{Z}$

$H_*(C) = 0 \Rightarrow H_*(C) \otimes H_*(D) = 0$, but

$C \otimes D = (\mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}) \Rightarrow H_2 \neq 0$

Ex $C = D = (\mathbb{Z} \xrightarrow{2} \mathbb{Z})$

$H_1(C) = 0 \Rightarrow H_1(C) \otimes H_0(D) = 0$

$C \otimes D = (\mathbb{Z} \xrightarrow{(2,-2)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(2,2)} \mathbb{Z}) \Rightarrow H_1 \neq 0.$

Proof of thm Define chain complexes

Z, B, H by

$$Z_n = Z_n(D)$$

$$B_n = B_n(D)$$

$$H_n = H_n(C)$$

and all three differentials trivial. The differential in D defines a chain map $\partial : D \to B$, and the sequence

$$0 \to Z \xrightarrow{\iota} D \xrightarrow{\partial} B \to 0$$

is exact. Choosing a basis for each $H_n(C)$ and cycle representatives, we obtain a chain map $\varphi : H \to C$, and, since C and H are free, the rows in the resulting commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & C \otimes Z \arrow[r] & C \otimes D \arrow[r] & C \otimes B \arrow[r] & 0 \\
0 \arrow[r] & H \otimes Z \arrow[r] \arrow[u] & H \otimes D \arrow[r] \arrow[u] & H \otimes B \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

are exact. By the five lemma, then, we may assume that the differential in D

is trivial. Since homology preserves direct sums, we may assume that $D = (\cdots \to 0 \to \underset{\uparrow r}{\mathbb{Z}} \to 0 \to \cdots)$, in which case

$$\bigoplus_{p+q=n} H_p(C) \otimes H_q(D) = H_{n-r}(C) \otimes \mathbb{Z}$$
$$\cong H_{n-r}(C)$$
$$\cong H_n(C \otimes D).$$

$\square$