Last time

- Kneser conjecture
- Goal: homology of products
- Tensor products of chain complexes
- Künneth theorem

Q How to relate $C_*(X \times Y)$ and $C_*(X) \otimes C_*(Y)$?

Corollary Let C and D be chain complexes of free Abelian groups. If both have the homology of a point, then so does $C \otimes D$.

Observation 1 If X and Y are contractible, then $C_*(X) \otimes C_*(Y)$ and $C_*(X \times Y)$ have the same homology. In particular, this is so if X and Y are standard simplices.

Observation 2 $C_0(X \times Y) \cong C_0(X) \otimes C_0(Y)$ canonically.

As we will show, the first implies that the second is all that matters.

In the following, we consider chain maps $\varphi_{X,Y}$ with fixed source and target $\{C_*(X)\otimes C_*(Y), C_*(X\times Y)\}$. The word "natural" refers to commutativity of (for example)

$$\begin{tikzcd}
C_*(X\times Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "(f\times g)_\#"'] & C_*(X)\otimes C_*(Y) \arrow[d, "f_\#\otimes g_\#"] \\
C_*(Z\times W) \arrow[r, "\varphi_{Z,W}"] & C_*(Z)\otimes C_*(W)
\end{tikzcd}$$

for all maps $f: X\to Z$ and $g: Y\to W$.

<u>Thm</u> (Acyclic models) Any two such natural chain maps are naturally chain homotopic provided they agree in degree 0.

<u>Proof</u> For concreteness, take

$$\varphi, \psi : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

(subscripts suppressed). We wish to define $\Gamma_n : C_n(X \times Y) \to \left(C_*(X) \otimes C_*(Y)\right)_{n+1}$ such that $\partial\Gamma + \Gamma\partial = \varphi - \psi$. Since $\varphi = \psi$

In degree 0, we may take $\Gamma_0 \equiv 0$ and proceed inductively. Naturality forces commutativity of

$$
\begin{tikzcd}
\sigma & C_*(X\times Y) \arrow[rr, "\Gamma"] & & C_*(X)\otimes C_*(Y) \\
\mathrm{id} \arrow[u, mapsto] & C_*(\Delta^n) \arrow[u, "\sigma_\#"] \arrow[r, "\Delta_\#"'] & C_*(\Delta^n\times\Delta^n) \arrow[ul, "(\pi_X\circ\sigma,\ \pi_Y\circ\sigma)_\#"'] \arrow[r, "\Gamma"] & C_*(\Delta^n)\otimes C_*(\Delta^n) \arrow[u, "(\pi_X\circ\sigma)_\#\otimes(\pi_Y\circ\sigma)_\#"']
\end{tikzcd}
$$

so Γ is determined by its value on

$$\alpha_n := \Delta_\#(\mathrm{id}_{\Delta^n}) \in C_n(\Delta^n\times\Delta^n).$$

Requiring Γ to be a chain homotopy forces the equation

$$\partial\Gamma(\alpha_n) \overset{?}{=} \varphi(\alpha_n) - \psi(\alpha_n) - \Gamma\partial(\alpha_n), \quad (\ast)$$

and we define $\Gamma(\alpha_n)$ to be <u>any</u> $(n+1)$-chain satisfying this equation. To check that such a chain exists, note that the RHS of $(\ast)$ is a cycle, since its boundary is

$$\varphi(\partial\alpha_n) - \psi(\partial\alpha_n) - \partial\Gamma\partial(\alpha_n)$$

and $\partial\Gamma\partial(\alpha_n) = \varphi(\partial\alpha_n) - \psi(\partial\alpha_n) - \Gamma\cancelto{0}{\partial^2(\alpha_n)}$ by induction. But $n>0$, so

$$H_n(C_*(\Delta^n)\otimes C_*(\Delta^n)) = 0$$

by the corollary, so every n-cycle is a boundary. $\square$

An identical argument shows that such a natural chain map exists; we simply replace the equation $\partial\Gamma = \varphi - \psi - \Gamma\partial$ with $\partial\varphi = \varphi\partial$.

<u>Corollary</u> $C_*(X\times Y)$ and $C_*(X)\otimes C_*(Y)$ are naturally chain homotopy equivalent.

<u>Proof</u> Choose natural chain maps

$$C_*(X\times Y) \xrightarrow{\varphi} C_*(X)\otimes C_*(Y) \xrightarrow{\psi} C_*(X\times Y)$$

extending the isomorphism in degree 0. Then $\psi\circ\varphi$ is a natural chain map restricting to the identity in degree 0, hence chain homotopic to the identity, and similarly for $\varphi\circ\psi$. $\square$

For later use, we record a specific choice.

<u>Def</u> The <u>Alexander-Whitney map</u> is the map

$$C_n(X \times Y) \longrightarrow \bigoplus_{p+q=n} C_p(X) \otimes C_q(Y)$$

$$\sigma \longmapsto \sum_{p+q=n} \pi_X \circ \sigma\big|_{[e_0,\dots,e_p]} \otimes \pi_Y \circ \sigma\big|_{[e_p,\dots,e_n]}$$

<u>Exercise</u> The AW map is a chain map.

Since this map is the canonical isomorphism for $n=0$, it is a chain homotopy equivalence by acyclic models.

Using the equivalence $C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$ and Künneth, we can now easily compute the homology of, say, an arbitrary product of spheres... but we could have done this before. Its true utility is hidden behind duality.

<u>Recall</u> The <u>dual</u> of the Abelian group A is the Abelian group

$$A^\vee := \mathrm{Hom}(A, \mathbb{Z})$$

of homomorphisms $\lambda: A \to \mathbb{Z}$, with group law

$$(\lambda+\mu)(a) = \lambda(a)+\mu(a).$$

<u>Ex</u> (1) $\mathbb{Z}^\vee \cong \mathbb{Z}$ via $\lambda \mapsto \lambda(1)$

(2) $(A \oplus B)^\vee \cong A^\vee \oplus B^\vee$ via $\lambda \mapsto (\lambda|_A, \lambda|_B)$

(3) So $(\mathbb{Z}^r)^\vee \cong \mathbb{Z}^r$

(4) But $\left(\bigoplus_{\mathbb{N}} A_n\right)^\vee \cong \prod_{\mathbb{N}} A_n^\vee \not\cong \bigoplus_{\mathbb{N}} A_n^\vee$.

(5) $(\mathbb{Z}/n\mathbb{Z})^\vee = 0$.

<u>Warning</u> There are other duals and other notations.

A homomorphism $\varphi: A \to B$ induces a homomorphism $\varphi^\vee: B^\vee \to A^\vee$ by restriction.

What kind of thing is the dual of a chain complex?

Def A <u>cochain complex</u> is a sequence $\{C^n\}_{n\geq 0}$ of Abelian groups, together with homomorphisms $\partial^n: C^{n-1} \to C^n$ such that $\partial^{n+1} \circ \partial^n = 0$.
The n^{th} <u>cohomology</u> group of (C, ∂) is

$$H^n(C) = \frac{\ker \partial^{n+1}}{\operatorname{im} \partial^n}$$

<u>Ex</u> If (C, ∂) is a chain complex, then $(C^\vee, \partial^\vee)$ is a cochain complex.

<u>Def</u> The complex of <u>singular cochains</u> on X is $C^*(X) = C_*(X)^\vee$. Its cohomology is the <u>singular cohomology</u> of X, denoted $H^*(X)$.