Last time

- Acyclic models
- $C_*(X\times Y) \simeq C_*(X)\otimes C_*(Y)$
- Duals and cochain complexes
- Cohomology

Recall that $H^*(X)$ is the cohomology of the cochain complex $C_*(X)^\vee$. More generally, given an Abelian group G, cohomology with <u>coefficients</u> in G is the cohomology of the cochain complex

$$C^n(X;G) := \mathrm{Hom}_{\mathbb{Z}}(C_n(X), G)$$

$$\partial\eta(\sigma) = \eta(\partial\sigma).$$

<u>Q</u> How different could this possibly be?

<u>A!</u> Not very.

The cochain map $f_\#^\vee$ induces a homomorphism $f^*: H^*(Y;G) \to H^*(X;G)$ and the key features of homology carry over unchanged.

(1) functoriality

(2) homotopy invariance

(3) subdivision

(4) Mayer-Vietoris / excision

(5) cellular cohomology

Ex The cellular cochain complex of $\mathbb{RP}^m$ is

$$\left(\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \right)^{\vee} \cong \left(\cdots \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0 \right)$$

$$\Longrightarrow H^n(\mathbb{RP}^m) \cong \begin{cases} \mathbb{Z} & n=0 \text{ or } n=m \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0<n<m \text{ even} \end{cases}$$

A2 Not at all.

Thm (Universal coefficients) Let C be a chain complex. The assignment

$$\begin{aligned} H^n(C^\vee) &\longrightarrow H_n(C)^\vee \\ [\lambda] &\longmapsto \big([c] \mapsto \lambda(c)\big) \end{aligned}$$

is a well-defined homomorphism. If C and $H_*(C)$ are free Abelian, it is an isomorphism.

The first claim is an easy exercise, and the second is almost identical to our proof of the Künneth theorem.

__Rmk__ The statement can be modified using Ext to be true in general.

__Cor__ If $H_*(X)$ is free of finite rank in each degree, then $H_*(X) \cong H^*(X)$ non-canonically.

A3 Completely!

Observation For Abelian groups A and B, there is a canonical homomorphism

$$A^\vee \otimes B^\vee \longrightarrow (A \otimes B)^\vee$$
$$\lambda \otimes \mu \longmapsto (a \otimes b \mapsto \lambda(a)\mu(b)).$$

For a chain complex C, there descend to a homomorphism $H^*(C^\vee) \otimes H^*(C^\vee) \xrightarrow{m} H^*((C \otimes C)^\vee)$.

<u>Def</u> The <u>cup product</u> is the homomorphism

$$H^*(X) \otimes H^*(X) \to H^*(X)$$
$$\alpha \otimes \beta \longmapsto \alpha \cup \beta = \alpha\beta$$

given by the composite

$$\begin{array}{l} H^*(X) \otimes H^*(X) \\ \quad \parallel \\ H^*(C_*(X)^\vee) \otimes H^*(C_*(X)^\vee) \xrightarrow{m} H^*\big((C_*(X) \otimes C_*(X))^\vee\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \cong \text{ dual A-W (e.g.)} \\ \qquad\qquad\qquad\qquad\qquad\qquad H^*(C_*(X \times X)^\vee) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel \\ \qquad\qquad\qquad\qquad\qquad\qquad H_*(X \times X) \xrightarrow{\Delta^*} H^*(X) \end{array}$$

<u>Exercise</u> For $\lambda \in H^p(X)$, $\mu \in H^q(X)$, $\lambda \cup \mu$ is the class of

$$\sigma \mapsto \tilde{\lambda}(\sigma|_{[e_0, \ldots, e_p]})\tilde{\mu}(\sigma|_{[e_p, \ldots, e_q]})$$

("front face/back face")

Def A **graded ring** is a ring R with a direct sum decomposition $R = \bigoplus_{n \geq 0} R_n$ of Abelian groups such that $R_m \cdot R_n \subseteq R_{m+n}$. We say that R is **graded commutative** if

$$r_1 r_2 = (-1)^{mn} r_2 r_1$$

for $r_1 \in R_m$ and $R_2 \in R_n$. A **graded ring homomorphism** between graded rings R and S is a ring homomorphism $\varphi : R \to S$ such that $\varphi(R_n) \subseteq S_n$.