Last time

- Determinants & row operations
- Alternating & multilinear properties
- Determinants detect invertibility

The relation to row operations also gives an effective method of computation.

Ex Let's calculate the determinant of

$$A = \begin{bmatrix} 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[1+2]{\text{swap}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[]{\text{Subtract } 1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[]{\text{swap } 3 \,\&\, 4} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix} = B$$

So
$$\begin{aligned} \det(A) &= (-1)^2 \det(B) \\ &= (-1)^2 \cdot 1 \cdot 7 \cdot (-1)(-2) \\ &= 14. \end{aligned}$$

Calculating $\det(A)$

(1) Use row operations to turn A into a matrix B such that $\det(B)$ is easy (e.g., triangular).

(2) If you used s many row swaps and divided rows by scalars $k_1, \ldots, k_r$, then

$$\det(A) = (-1)^s k_1 \cdots k_r \det(B)$$

Using this technique, we can calculate the determinant of a product. It's easy to see that, if A' is obtained from A using a row operation, then $A'B$ is obtained from AB using the same row operation. If A is invertible, then, we have

$$[A \mid I_n] \longrightarrow [I_n \mid A^{-1}]$$

using s swaps and dividing by $k_1, \ldots, k_r$. Therefore,

$$[AB \mid B] \longrightarrow [B \mid A^{-1}B]$$

using the same operations. Then

$$\det(AB) = (-1)^s k_1 \cdots k_r \det(B)$$
$$= (-1)^s k_1 \cdots k_r \det(I_n)\det(B)$$
$$= \det(A)\det(B).$$

If A is not invertible, then neither is AB; indeed, since A is $n \times n$

$$A \text{ not invertible} \iff \mathrm{im}(A) \neq \mathbb{R}^n,$$

but $\mathrm{im}(AB)$ is contained in $\mathrm{im}(A)$, so $\mathrm{im}(AB) \neq \mathbb{R}^n$. In this case,

$$\det(AB) = 0 = 0 \cdot \det(B) = \det(A)\det(B)$$

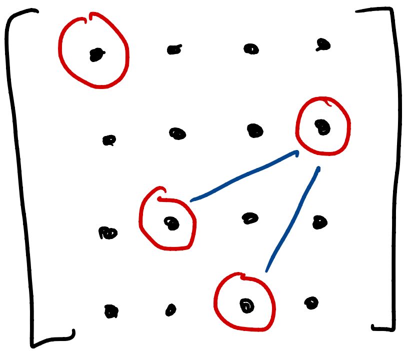
$$\boxed{\det(AB) = \det(A)\det(B)}$$

From this, it follows that

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

Finally, we record the relationship to transposes:

$$\boxed{\det(A) = \det(A^T)}$$



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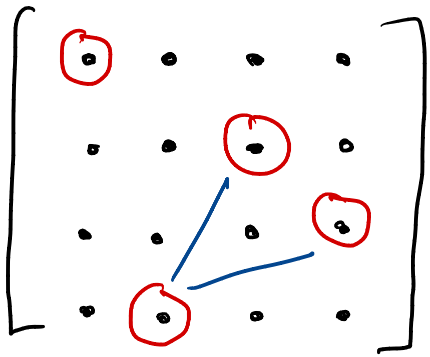

P^T

$\mathrm{prod}(P) = \mathrm{prod}(P^T) \qquad \mathrm{sign}(P) = \mathrm{sign}(P^T)$

We return to Sarrus' rule:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$- \quad - \quad - \qquad\qquad + \quad + \quad +$$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$+ a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$ (to be continued)