Last time

- Calculating $\det(A)$ by row reduction
- Algebraic properties of determinants

Last time, we saw that, for A 3×3,

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},$$

a <u>recursive</u> formula for the 3×3 determinant in terms of the 2×2 determinant.

Laplace expansion Let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by omitting the i^{th} row and j^{th} column from A.

(1) Expansion along row i:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

(2) Expansion down column j:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Ex

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 9 & 1 & 3 & 0 \\ 9 & 2 & 2 & 0 \\ 5 & 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Expanding down the 2^{nd} column:

$$\det(A) = \cancel{-0 \cdot \det \begin{bmatrix} 9 & 3 & 0 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}}^{\,0} + 1 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$$

$$- 2 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 3 & 0 \\ 5 & 0 & 3 \end{bmatrix} + \cancel{0 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 9 & 3 & 0 \\ 9 & 2 & 0 \end{bmatrix}}^{\,0}$$

$$= 5 \det \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 1 \\ 9 & 2 \end{bmatrix}$$

$$- 2 \left(2 \det \begin{bmatrix} 9 & 3 \\ 5 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 1 \\ 9 & 3 \end{bmatrix} \right)$$

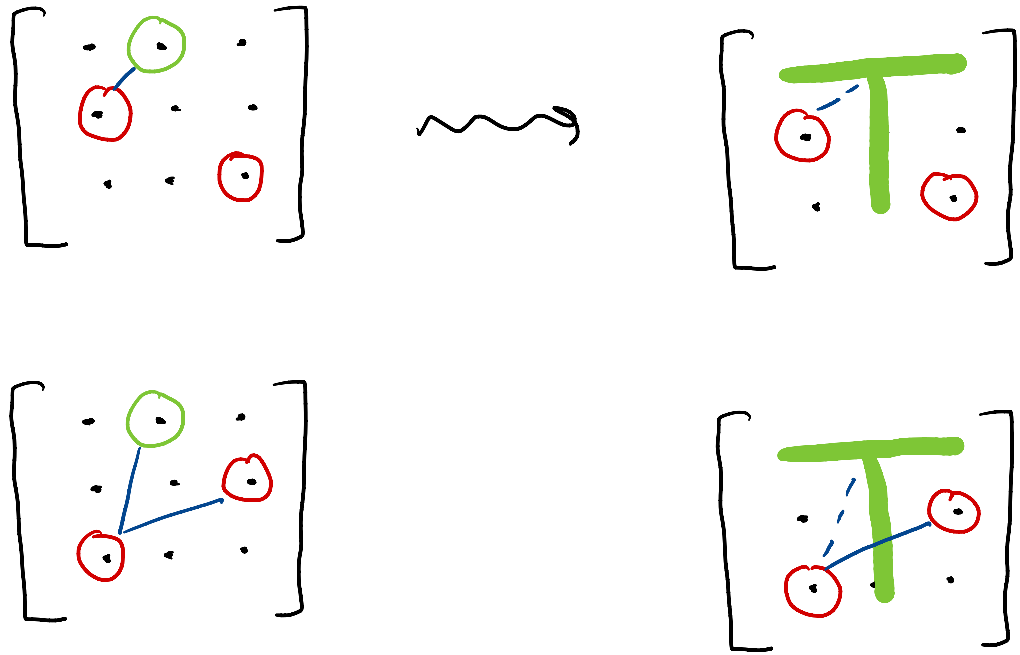
$$= 5(-4) + 3(-7) - 4(-15) - 6(-6)$$

$$= -20 - 21 + 60 + 36$$

$$= 55$$

Why does this work? For every entry in the $i^{\underline{th}}$ row, collect all the patterns involving that entry, and factor out that entry to get the determinant of the smaller matrix, up to a sign.

accounting for the lost inversions.



Diagonal matrices are good because

(1) they're easy to work with

(2) their geometric meaning is transparent

(3) the systems of equations they represent are "uncoupled", so easy to solve

Matrices that are similar to diagonal matrices are just as good.

<u>Def</u> An $n \times n$ matrix A is <u>diagonalizable</u> if A is similar to a diagonal matrix.

In other words, $S^{-1}AS$ is diagonal for some invertible $n \times n$ matrix S, which is to say there is a basis $\mathcal{B}$ (the columns of S) for $\mathbb{R}^n$ such that the $\mathcal{B}$-matrix of A is diagonal.

What does this mean, exactly?

Suppose

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \qquad S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = D$$

In this case,

$$\begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix} = AS = SD = \begin{bmatrix} \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \end{bmatrix}.$$

<u>Def</u> An <u>eigenvector</u> of A is a vector $\vec{v} \neq \vec{0}$ such that

$$A\vec{v} = \lambda\vec{v}$$

for some scalar λ, called the <u>eigenvalue</u> associated to $\vec{v}$. A basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ is called an <u>eigenbasis</u> for A if each $\vec{v}_i$ is an eigenvector of A.

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diagonalizable $\Longleftrightarrow$ an eigenbasis exists

More specifically, if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly independent eigenvectors of A

with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

<u>Ex</u> Every nonzero vector is an eigenvector for I_n with eigenvalue 1.

<u>Ex</u> Any nonzero vector in V is an eigenvector for proj_V with eigenvalue 1.
Any nonzero vector in $V^\perp$ is an

eigenvector with eigenvalue 0.

<u>Ex</u> An eigenvector with eigenvalue 0 is the same thing as a nonzero vector in $\ker(A)$.