Last time

- Calculating eigenvalues
- Characteristic equation $\det(A-\lambda I_n)=0$

In the examples so far, $\det(A-\lambda I_n)$ is a polynomial of degree n. This is true in general, since $\mathrm{Prod}(P)$ is a polynomial of degree $\leq n$, with equality for the diagonal pattern.

Def The characteristic polynomial of A is $f_A(\lambda) = \det(A - \lambda I_n)$.

This polynomial has

- degree n
- leading coefficient $(-1)^n$
- second coefficient $(-1)^{n-1}\operatorname{tr}(A)$
- constant coefficient $\det(A)$.

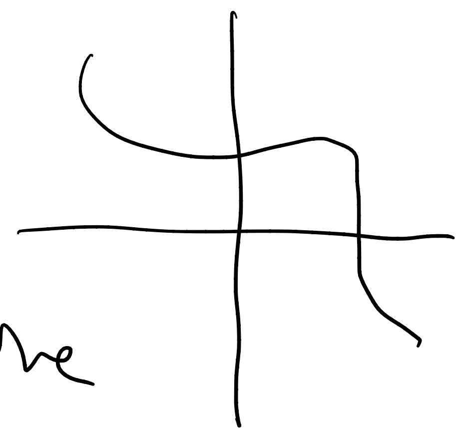
The eigenvalues of A are the roots of $f_A(\lambda)$.

The fact that $f_A(\lambda)$ is a polynomial helps us count eigenvalues.

(1) If n is odd, then A has at least one eigenvalue. Indeed,

$$\lim_{\lambda \to \pm\infty} f_A(\lambda) = \lim_{\lambda \to \pm\infty} (-1)^n \lambda^n$$

$$= \mp \infty$$



So the graph of $f_A(\lambda)$ crosses the λ axis by the Intermediate value theorem.

(2) An eigenvalue might occur "with multiplicity."

<u>Def</u> If λ_0 is an eigenvalue of A, the <u>algebraic multiplicity</u> of λ_0 is the natural number k such that

$$f_A(\lambda) = (\lambda_0 - \lambda)^k g(\lambda)$$

where $g(\lambda)$ is a polynomial with $g(\lambda_0) \neq 0$.

Ex

If $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$, then

$f_A(\lambda) = (1-\lambda)^3(2-\lambda)^2$, so $\lambda = 1$ has algebraic multiplicity 3 and $\lambda = 2$ has algebraic multiplicity 2.

So an $n \times n$ matrix has <u>at most</u> n eigenvalues, counted with multiplicity.

If A has n eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct), so that $f_A(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$, then

$$tr(A) = \lambda_1 + \cdots + \lambda_n$$

$$\det(A) = \lambda_1 \cdots \lambda_n .$$

We now have a good strategy for determining the eigenvalues of an $n \times n$ matrix A: solve the characteristic equation $\det(A - \lambda I_n) = 0$.

What about eigenvectors?

<u>Def</u> The subspace spanned by the eigenvectors of A with eigenvalue λ is called the <u>eigenspace</u> associated to λ, i.e.,

$$E_\lambda = \ker(A - \lambda I_n).$$

Every nonzero vector in E_λ is an eigenvector with eigenvalue λ (and vice versa).