Last time

- Characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_n)$
- Algebraic multiplicity
- Eigenspaces $E_\lambda = \ker(A - \lambda I_n)$

<u>Ex</u> As we saw, the eigenvalues of $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are $\lambda = 5, -1$, and

$$E_5 = \ker \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$E_{-1} = \ker \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

Ex Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Since A is upper triangular, its eigenvalues are $\lambda = 0, 1$. To find the eigenspaces,

$$E_1 = \ker(A - I_3) = \ker \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$E_0 = \ker(A - 0 \cdot I_3) = \ker\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \ker\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$$

Since E_1 and E_0 are both 1 dimensional, it is only possible to find 2 linearly independent eigenvectors for A, so there is no eigenbasis, i.e., A is not diagonalizable.

Def The geometric multiplicity of λ_0 as an eigenvalue of A is the dimension of the eigenspace associated to λ_0 ($\dim E_{\lambda_0}$).

In the last example, we had $f_A(\lambda) = \lambda(1-\lambda)^2$, so:

λ	alg. mult.	geom. mult.
0	1	1
1	2	1 $\leftarrow$!

The following three conditions are equivalent for an $n \times n$ matrix A.

(1) A is diagonalizable

(2) A has n eigenvalues (not necessarily distinct) whose algebraic and geometric multiplicities coincide

(3) The dimensions of the eigenspaces of A add up to n.

The main ingredient here is:

$$\boxed{\text{Eigenvectors for distinct eigenvalues are linearly independent}}$$

Why? Suppose $\vec{v}_1, \ldots, \vec{v}_r$ are eigenvectors for distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. If $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly dependent, then there is a <u>first</u> redundant vector $\vec{v}_i$.

Since $\vec{v}_i$ is redundant,

$$\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1}.$$

Applying the matrix $A - \lambda_i I_n$ gives

$$0 = (A-\lambda_i I_n)\vec{v}_i = c_1(A-\lambda_i I_n)\vec{v}_1 + \cdots$$

$$= c_1\left[A\vec{v}_i - \lambda_1\vec{v}_1\right] + \cdots$$

$$= c_1\left[\lambda_i\vec{v}_i - \lambda_i\vec{v}_1\right] + \cdots$$

$$= c_1(\lambda_i - \lambda_1)\vec{v}_1 + \cdots$$

$$\Rightarrow c_1(\lambda_i - \lambda_1)\vec{v}_1 + \cdots + c_{i-1}(\lambda_i - \lambda_{i-1})\vec{v}_{i-1} = 0$$

This is linearly relation among the first $i-1$ vectors, so $\vec{v}_i$ wasn't the <u>first</u> redundant vector after all!

If A has n <u>distinct</u> eigenvalues, then A is diagonalizable

Diagonalizing a matrix A

(1) Solve the characteristic equation to find the eigenvalues of A

(2) For each eigenvalue, find a basis for $E_\lambda = \ker(A - \lambda I_n)$.

(3) If the dimensions of the eigenspaces do not add up to n, then A is not diagonalizable.

(4) Otherwise, $S^{-1}AS = B$, where

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \qquad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

the $\vec{v}_i$ are the basis vectors from step (2), and λ_i is the eigenvalue of $\vec{v}_i$.

<u>Ex</u> $A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$f_A(\lambda) = \det\begin{bmatrix} 1-\lambda & 0 & 0 \\ -4 & -\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\det\begin{bmatrix} -\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix}$$

$$= -\lambda(1-\lambda)^2 \qquad \lambda = 0, 1$$

$$E_0 = \ker\begin{bmatrix} 1 & 0 & 0 \\ -4 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$E_1 = \ker\begin{bmatrix} 0 & 0 & 0 \\ -4 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \ker\begin{bmatrix} -4 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-4x_1 - x_2 + 2x_3 = 0$

x_2, x_3 free

$$= \operatorname{Span}\left(\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right)$$

λ	alg. mult.	geom. mult.
0	1 =	1
1	2 =	2
		3 ✓

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$