

# Last time

- Diagonalization algorithm

Q How badly can diagonalizability fail?

Ex  $A = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$  The only eigenvalue is  $\lambda = 5$

$$E_5 = \ker \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda$	alg. mult.	geom. mult.
5	4	1

Similarly, the  $n \times n$  matrix

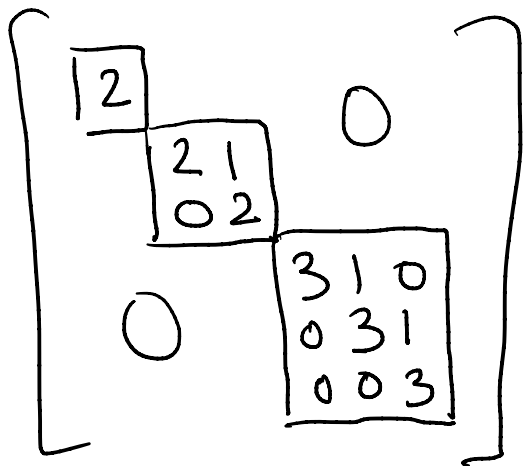
$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & 0 & \ddots & \lambda \\ & & & \lambda \end{bmatrix}$$

has eigenvalue  $\lambda$  with algebraic multiplicity  $n$  and geometric multiplicity 1.

Combining these, we can build custom non-diagonalizable matrices.

Ex A  $6 \times 6$  matrix with

$\lambda$	alg. mult.	geo. mult.
2	3	2
3	3	1



A last word: similar matrices have the same characteristic polynomial, the same eigenvalues with the same algebraic and geometric multiplicities, and the same determinant and trace. Most of this comes down to the fact that

$$A = SBS^{-1} \Rightarrow A - \lambda I_n = S(B - \lambda I_n)S^{-1}.$$

But similar matrices don't necessarily have the same eigenvectors.

Ex We've seen many times now that projection onto the line  $L$  parallel to  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$  is similar to projection onto the  $x$ -axis  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ .

$\lambda$	$E_\lambda$ for $\text{proj}_L$	$E_\lambda$ for $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
1	$\text{Span}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$	$\text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$
0	$\text{Span}\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right)$	$\text{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

We've spent lots of time investigating two types of special basis: orthonormal bases and eigenbases.

Q When can we have both?

Def We say  $A$  is orthogonally diagonalizable if there exists an orthonormal eigenbasis for  $A$ , i.e., if there is an orthogonal matrix  $S$  such that  $S^{-1}AS$  is diagonal.

Ex Diagonal matrices

Ex Orthogonal projection

Ex Reflection

All three types are symmetric, and this is no coincidence if  $S^{-1}AS$  is diagonal, then

$$\begin{aligned} S^{-1}AS &= (S^{-1}AS)^T \\ &= S^T A^T (S^{-1})^T \\ &= \bar{S}^{-1} A^T (S^T)^T \\ &= S^{-1} A^T S \quad \implies A = A^T \end{aligned}$$

The converse is also true.

Spectral theorem  $A$  is orthogonally diagonalizable if and only if  $A$  is symmetric.

Ex  $A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$   $f_A(\lambda) = \lambda^2 - 11\lambda + 24$

$$\lambda = (\lambda - 3)(\lambda - 8)$$

$$E_3 = \ker \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \text{Span} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$E_8 = \ker \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = \ker \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \text{Span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$



Since  $E_3$  is orthogonal to  $E_2$ , we get an ONEB by choosing a unit vector from each:

$$\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

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This example illustrates a general fact.

If  $A$  is symmetric, then eigenvectors for distinct eigenvalues are orthogonal

Why? Suppose  $\vec{v}_1$  and  $\vec{v}_2$  are eigenvectors for  $\lambda_1$  and  $\lambda_2$ , respectively.

We'd like to say that  $\vec{v}_1 \cdot \vec{v}_2 = 0$ . We have that

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1 \cdot A \vec{v}_2 = \vec{v}_1 \cdot \lambda_2 \vec{v}_2 = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2).$$

But, on the other hand,

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T A^T \vec{v}_2 = (A \vec{v}_1)^T \vec{v}_2 = (\lambda_1 \vec{v}_1)^T \cdot \vec{v}_2 = \lambda_1 (\vec{v}_1 \cdot \vec{v}_2).$$

Subtracting these, we have

$$(\lambda_1 - \lambda_2) (\vec{v}_1 \cdot \vec{v}_2) = 0.$$

Since  $\lambda_1 \neq \lambda_2$ ,  $\vec{v}_1 \cdot \vec{v}_2 = 0$ .

So far, this shows that a symmetric matrix is orthogonally diagonalizable if it is diagonalizable. It remains to show that symmetric matrices are diagonalizable, which we omit.