Last time

- Singular value decomposition
- Quadratic forms + symmetric matrices

Quadratic form	Matrix
x^2+y^2	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
x^2-y^2	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$8x^2-4xy+5y^2$	$\begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$
$3x^2-2xy+y^2-6xz$	$\begin{bmatrix} 3 & -1 & -3 \\ -1 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}$

Ex $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$, $q_A = 8x^2 - 4xy + 5y^2$.

$$f_A(\lambda) = \lambda^2 - 13\lambda + 36$$
$$= (\lambda - 9)(\lambda - 4) \qquad \lambda = 4, 9$$

$$E_4 = \ker \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} = \ker \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$E_9 = \ker \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$A = SDS^{-1}, \quad S = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$
$$= [\vec{u}_1, \vec{u}_2]$$

$$q_A(c_1\vec{u}_1 + c_2\vec{u}_2) = (c_1\vec{u}_1 + c_2\vec{u}_2)^T A (c_1\vec{u}_1 + c_2\vec{u}_2)$$

$$= c_1^2 \vec{u}_1^T A \vec{u}_1 + c_1 c_2 \vec{u}_1^T A \vec{u}_2 + c_2 c_1 \vec{u}_2^T A \vec{u}_1 + c_2^2 \vec{u}_2^T A \vec{u}_2$$

$$= 4c_1^2 \vec{u}_1^T u_1 + 9c_1 c_2 \vec{u}_1^T \vec{u}_2 + 4c_2 c_1 \vec{u}_2^T \vec{u}_1 + 9c_2^2 \vec{u}_2^T \vec{u}_2$$

$$= 4c_1^2 + 9c_2^2.$$

We can do even better! Setting $\vec{v}_1 = \frac{1}{2}\vec{u}_1$ and $\vec{v}_2 = \frac{1}{3}\vec{u}_2$,

$$q_A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1^2 + c_2^2.$$

Given a quadratic form q, there is a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that

$$q(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = \sum_i \varepsilon_i c_i^2,$$

where ε_i is 1, -1, or 0.

(1) Write $q = q_A$ with A symmetric

(2) Find an ONEB $\{\vec{u}_1, \ldots, \vec{u}_n\}$ for A.

(3) Define

$$\vec{v}_i = \begin{cases} \frac{1}{\sqrt{|\lambda_i|}}\vec{u}_i & \text{if } \lambda_i \neq 0 \\ \vec{u}_i & \text{if } \lambda_i = 0 \end{cases}.$$

<u>Ex</u> $q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xy$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$q\left(c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = q\left(\begin{bmatrix} c_1+c_2 \\ c_1-c_2 \end{bmatrix}\right)$$

$$= (c_1+c_2)(c_1-c_2)$$

$$= c_1^2 - c_2^2.$$

We introduce terminology for some properties of a quadratic form q.

Name	Property of q	Property of A
Positive definite	$q(\vec{x}) > 0$ if $\vec{x} \neq 0$	All eigenvalues positive
Positive semidefinite	$q(\vec{x}) \geq 0$	All eigenvalues non-negative
Indefinite	positive and negative values	positive and negative eigenvalues

Fact q_A is positive definite if and only if the determinants of

$$[a_{11}], \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \ldots, \quad A$$

are all positive.

Ex $q\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 5x^2 - 4xy + 2y^2 + 2xz + 2yz + 2z^2$

$$A = \begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\det[5] = 5 > 0 \checkmark$$

$$\det\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = 10 - 4 = 6 > 0 \checkmark$$

$$\det\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = -\det\begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & -7 & -9 \end{bmatrix}$$

$$= -4\det\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5/4 \\ 0 & 0 & -1/4 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 1 > 0 \checkmark$$

q is positive definite