Last time

- Quadratic forms + eigenvalues
- Definiteness

Ex Suppose the populations of coyotes and rabbits are given as a function of time t (in years, say) by

$$c(t+1) = 0.86c(t) + 0.08r(t)$$
$$r(t+1) = -0.12c(t) + 1.14r(t)$$

which we can write as $\vec{x}_{t+1} = A\vec{x}_t$,

where

$$\vec{x}_t = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix} \qquad A = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}.$$

So the population at time t is determined by the initial population by

$$\vec{x}_t = A^t \vec{x}_0$$

$$\vec{x}_0 \xrightarrow{A} \vec{x}_1 \xrightarrow{A} \vec{x}_2 \xrightarrow{A} \cdots$$

We'd like an explicit formula for $\vec{x}_t$ as a function of t. For some initial conditions, this is easy.

(1) If $\vec{x}_0 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$, then

$$\vec{x}_1 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 110 \\ 330 \end{bmatrix} = 1.1 \begin{bmatrix} 100 \\ 300 \end{bmatrix}$$

$$\Longrightarrow \vec{x}_1 = 1.1 \vec{x}_0$$

$$\Longrightarrow \vec{x}_t = (1.1)^t \vec{x}_0,$$

i.e., $c(t) = 100(1.1)^t$ and $r(t) = 300(1.1)^t$.

Both populations grow exponentially at a rate of 10%.

(2) If $\vec{x}_0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$, then

$$\vec{x}_1 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix} = 0.9 \begin{bmatrix} 200 \\ 100 \end{bmatrix},$$

so, as before, $\vec{x}_t = (0.9)^t \vec{x}_0$, i.e.,

$c(t) = 200(0.9)^t$ and $r(t) = 100(0.9)^t$.

Both populations decay exponentially.

(3) If $\vec{x}_0 = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$, things don't work out as nicely. But

$$\vec{x}_0 = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = 2\begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4\begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

$$= 2\vec{v} + 4\vec{w}$$

$$\Longrightarrow \vec{x}_t = A^t \vec{x}_0$$

$$= A^t(2\vec{v} + 4\vec{w})$$

$$= 2A^t\vec{v} + 4A^t\vec{w}$$

$$= 2(1.1)^t\vec{v} + 4(0.9)^t\vec{w},$$

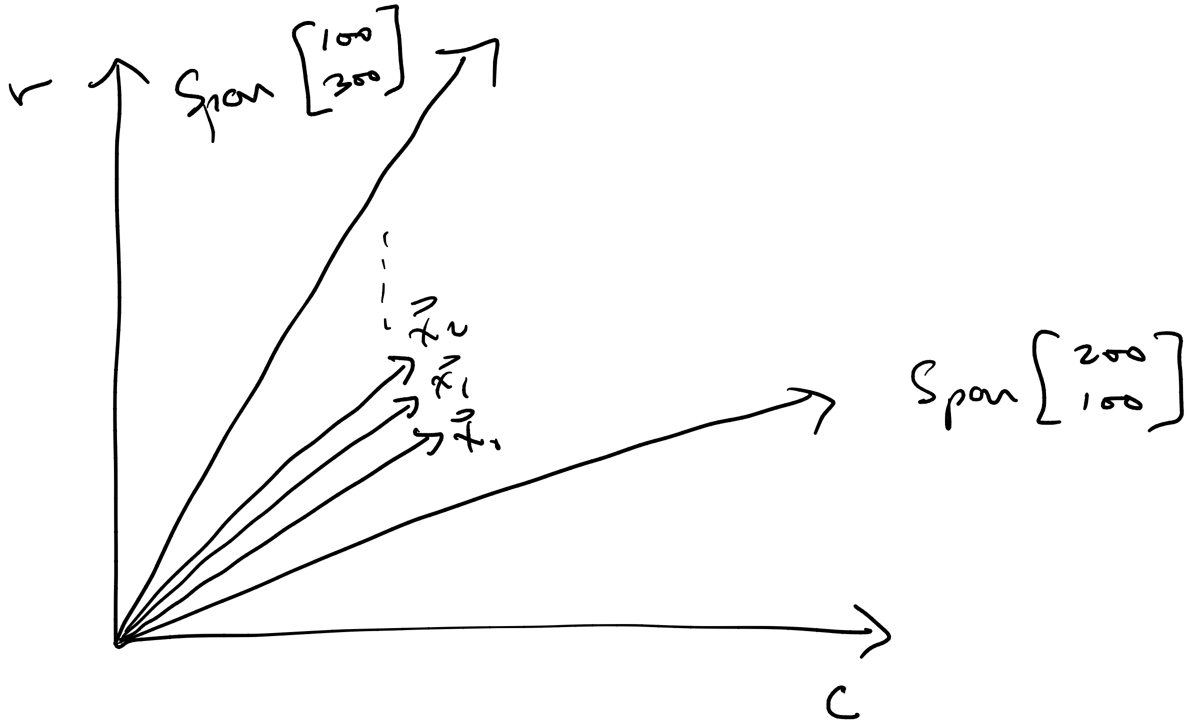
$$\Longrightarrow \begin{cases} c(t) = 200\,(1.1)^t + 800\,(0.9)^t \\ r(t) = 600\,(1.1)^t + 400\,(0.9)^t \end{cases}$$

As $t \to \infty$, both approach exponential growth by 10%, and the ratio of rabbits to coyotes approaches

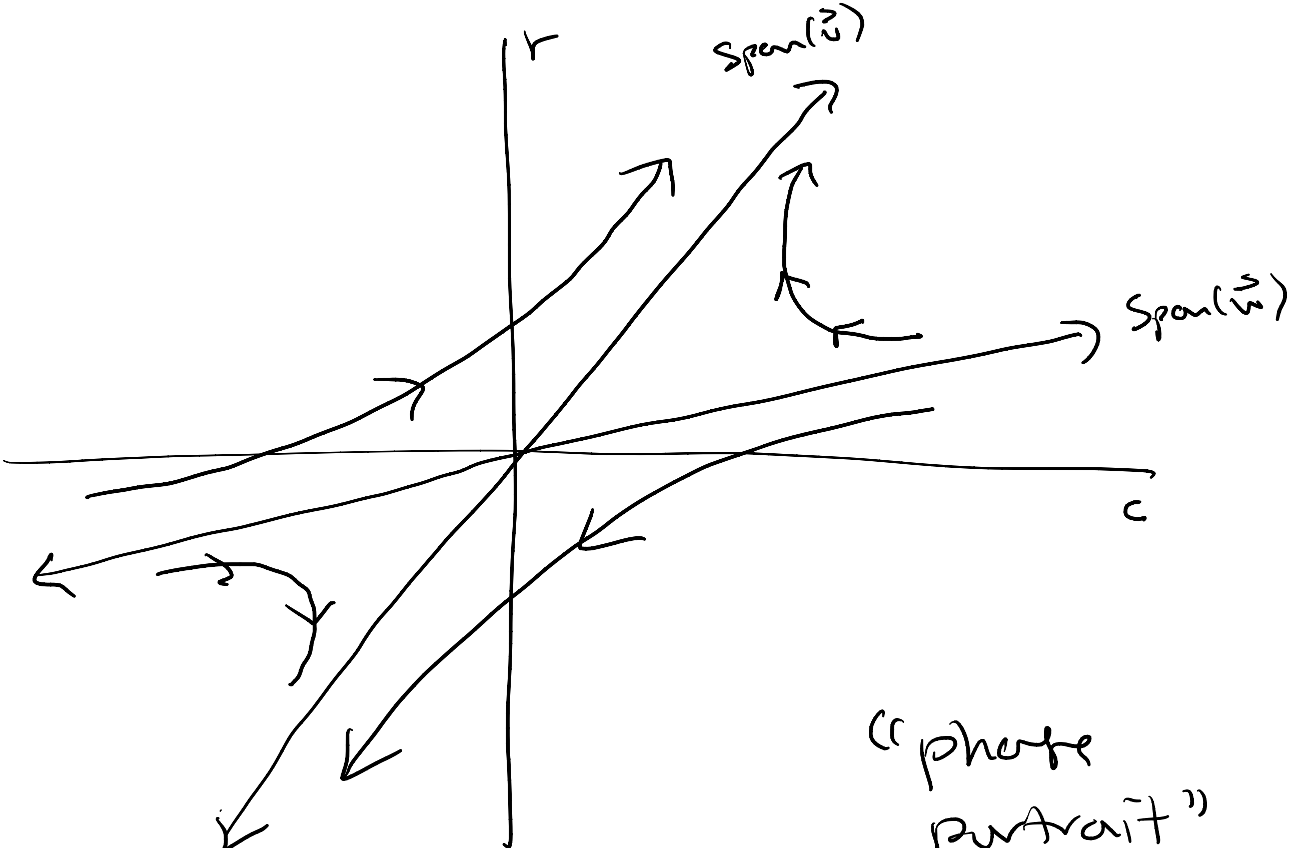
$$\lim_{t\to\infty} \frac{r(t)}{c(t)} = \lim_{t\to\infty} \frac{600\,(1.1)^t + 400\,(0.9)^t}{200\,(1.1)^t + 800\,(0.9)^t}$$

$$= \lim_{t\to\infty} \frac{600\,(1.1)^t}{200\,(1.1)^t}$$

$$= 3.$$



r
Span $\begin{bmatrix}100\\300\end{bmatrix}$
$\vec{x}_2$
$\vec{x}_1$
$\vec{x}_0$
Span $\begin{bmatrix}200\\100\end{bmatrix}$
c



("phase portrait")

The setup $\vec{x}_{t+1} = A\vec{x}_t$ is called a discrete dynamical system with initial value $\vec{x}_0$.

Ex

$$\begin{tikzcd} 1 \arrow[r, shift left] \arrow[d] & 2 \arrow[l, shift left] \arrow[dl] \\ 3 \arrow[r] & 4 \arrow[u] \end{tikzcd}$$

mini-internet

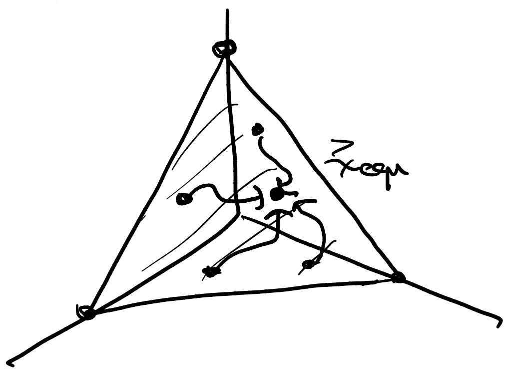
$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

transition matrix

This dynamical system models random web surfing. The relative influence of the pages is measured by the "equilibrium distribution", i.e., a distribution vector $\vec{x}_{equ}$ with

$$A\vec{x}_{equ} = \vec{x}_{equ}$$

We now recognize this as an eigenvector of A with eigenvalue $\lambda = 1$.



<u>Ex</u> Another transition matrix:

$$A = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}$$

To understand this dynamical system, let's diagonalize. The eigenvalues of A are $1/10$ of the eigenvalues of $\begin{bmatrix} 4 & 3 \\ 6 & 7 \end{bmatrix}$:

$$\det\begin{bmatrix} 4-\lambda & 3 \\ 6 & 7-\lambda \end{bmatrix} = (4-\lambda)(7-\lambda)-18$$
$$= \lambda^2 - 11\lambda + 10$$
$$= (\lambda-10)(\lambda-1)$$

So the eigenvalues of A are $\lambda = 1, 1/10$.

$$E_1 = \ker \frac{1}{10}\begin{bmatrix} -6 & 3 \\ 6 & -3 \end{bmatrix} = \ker\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$E_{1/10} = \ker \frac{1}{10}\begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{Span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So A is diagonalizable with

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1/10 \end{bmatrix}.$$

Then $A^t = (SBS^{-1})^t$

$$S^{-1} = \frac{1}{3} S$$

$$= S B^t S^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/10^t \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1/10^t \\ 2 & -1/10^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 + 2/10^t & 1 - 1/10^t \\ 2 - 2/10^t & 2 + 1/10^t \end{bmatrix}$$

The long term behavior is given by

$$\lim_{t\to\infty} A^t = \lim_{t\to\infty} \frac{1}{3}\begin{bmatrix} 1 + 2/10^t & 1 - 1/10^t \\ 2 - 2/10^t & 2 + 1/10^t \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

So, for example, if $\vec{x}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, then

$$\vec{x}_t = \frac{1}{3}\begin{bmatrix} 1 + 2/10^t & 1 - 1/10^t \\ 2 - 2/10^t & 2 + 1/10^t \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 + 1/10^t \\ 4 - 1/10^t \end{bmatrix} \qquad \lim_{t\to\infty} \vec{x}_t = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

DDS's by diagonalization $\vec{x}_t = A^t \vec{x}_0$

(1) Diagonalize A as $S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

(2) $A^t = S \begin{bmatrix} \lambda_1^t & & 0 \\ & \ddots & \\ 0 & & \lambda_n^t \end{bmatrix} S^{-1}$