Last time

- Orthogonal diagonalization algorithm
- Understanding A via A^TA
- Singular values $\sigma_i = \sqrt{\lambda_i}$, $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$

$$\boxed{\begin{array}{l}\text{For any } m\times n \text{ matrix } A\text{, there is a basis } \{\vec{v}_1, \ldots, \vec{v}_n\} \text{ such that}\\ \quad (1)\ \{\vec{v}_1, \ldots, \vec{v}_n\} \text{ is orthonormal}\\ \quad (2)\ \{A\vec{v}_1, \ldots, A\vec{v}_n\} \text{ is orthogonal}\\ \quad (3)\ |A\vec{v}_i| = \sigma_i .\end{array}}$$

As we've seen, the $\vec{v}_i$ are an orthonormal <u>eigenbasis</u> for A^TA.

Now, if A has rank r, then $A\vec{v}_i \neq 0$ for $1 \leq i \leq r$. So if we set

$$\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i,$$

then $\{\vec{u}_1, \ldots, \vec{u}_r\}$ is orthonormal. Expanding this to an ONB for $\mathbb{R}^m$, we have the equations

$$A\vec{v}_i = \begin{cases} \sigma_i \vec{u}_i & i = 1, \ldots, r \\ 0 & i = r+1, \ldots, m \end{cases}$$

or $AV = U\Sigma$, where

$$V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \qquad U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$$

$$\Sigma = \underbrace{\left.\begin{bmatrix} \begin{matrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}\right\}}_{n} m$$

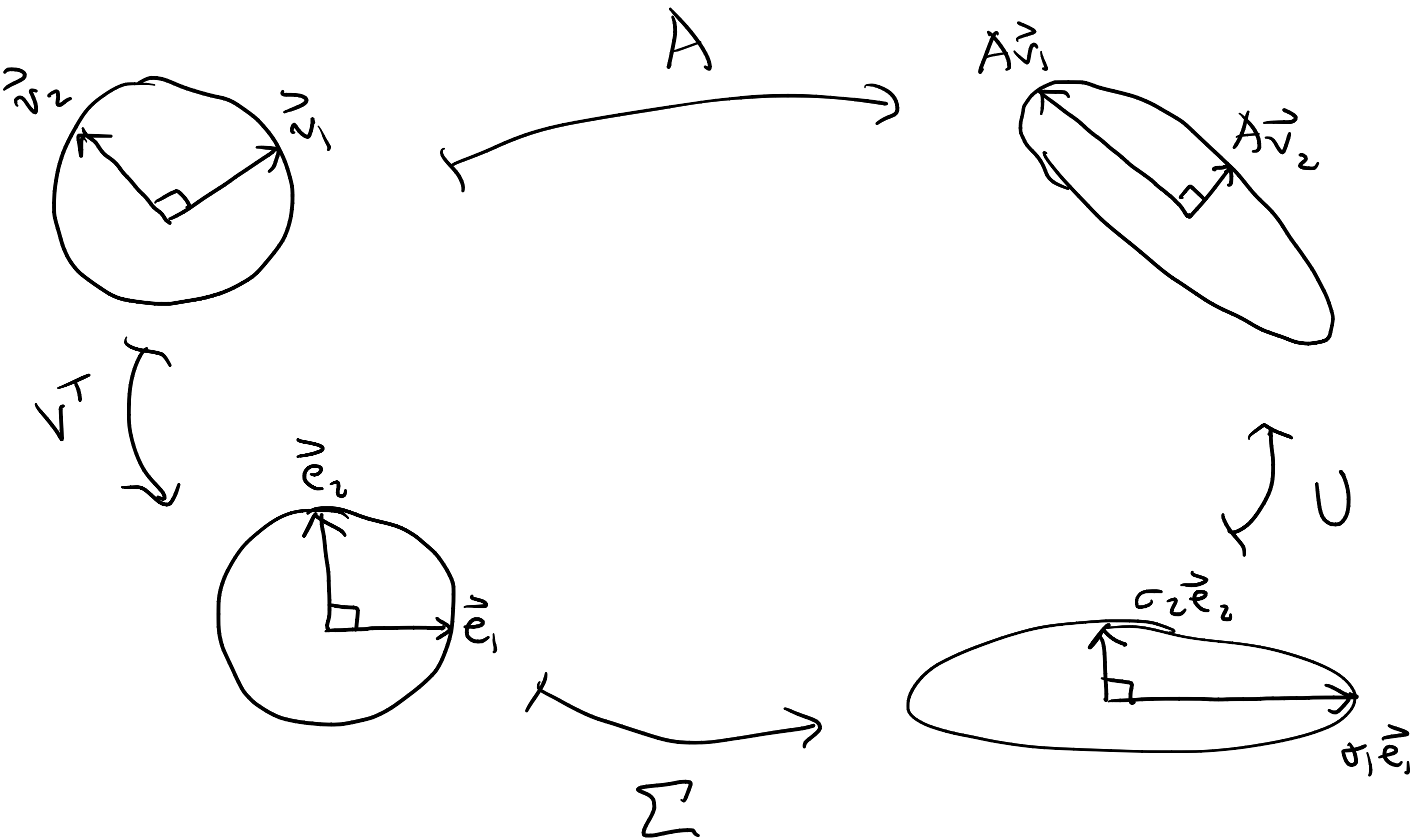
The matrices U and V are both orthogonal, so this equation becomes

$$A = U\Sigma V^T.$$

<u>Singular value decomposition</u> An $m \times n$ matrix A of rank r can be written as

$$A = U\Sigma V^T$$

where U is $m \times m$ orthogonal, V is $n \times n$ orthogonal, and Σ is $m \times n$ with the first r diagonal entries positive and all other entries 0.



$\vec{v}_2$
$\vec{v}_1$
A
$A\vec{v}_1$
$A\vec{v}_2$
V^T
U
$\vec{e}_2$
$\vec{e}_1$
$\sigma_2\vec{e}_2$
$\sigma_1\vec{e}_1$
Σ

To find SVD for A:

(1) Diagonalize A^TA to obtain ONB $\{\vec{v}_1, \dots, \vec{v}_n\}$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$.

(2) Set $\sigma_i = \sqrt{\lambda_i}$ and $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$ for $i = 1, \dots, r$.

(3) Expand $\{\vec{u}_1, \dots, \vec{u}_r\}$ to an ONB $\{\vec{u}_1, \dots, \vec{u}_m\}$.

(4) $V = [\vec{v}_1 \cdots \vec{v}_n]$

$U = [\vec{u}_1 \cdots \vec{u}_m]$

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]$$

Next, we apply the spectral theorem to *nonlinear* algebra.

Def A *quadratic form* is a function

$$q : \mathbb{R}^n \longrightarrow \mathbb{R}$$

of the form $q(\vec{x}) = \sum_{i \le j} c_{ij} x_i x_j$.

Ex $q(x,y) = x^2 + y^2$

Ex $q(x,y) = x^2 - y^2$

Ex $q(x,y) = 8x^2 - 4xy + 5y^2$

Ex $q(x,y,z) = 3x^2 - 2xy + y^2 - 6xz$

<u>Ex</u> Given a matrix A, the function

$$q_A(\vec{x}) = \vec{x}^T A \vec{x}$$

is a quadratic form. Indeed,

$$q_A(\vec{x}) = \sum_{i,j} x_i A_{ij} x_j$$

$$= \sum_{i \le j} c_{ij} x_i x_j,$$

where

$$c_{ij} = \begin{cases} A_{ij} & \text{if } i = j \\ A_{ij} + A_{ji} & \text{if } i \ne j. \end{cases}$$

Clearly, then, every quadratic form is determined by a matrix, and different matrices determine the same quadratic form. But $q = q_A$ for a unique symmetric matrix; we take

$$A_{ji} = A_{ij} = \begin{cases} c_{ij} & \text{if } i = j \\ c_{ij}/2 & \text{if } i \neq j. \end{cases}$$

Quadratic form	Matrix
$x^2 + y^2$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$x^2 - y^2$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$8x^2 - 4xy + 5y^2$	$\begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$
$3x^2 - 2xy + y^2 - 6xz$	$\begin{bmatrix} 3 & -1 & -3 \\ -1 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}$