Last time

- Cohomology w/ coefficients
- $H^*(X) \cong H_*(X)^\vee$ for $H_*(X)$ free
- Cup product
- Graded (commutative) rings $\quad R_m R_n \subseteq R_{m+n}$
  $\left(r_1 r_2 = (-1)^{mn} r_2 r_1 \right)$

<u>Ex</u> The tensor algebra $T(A) = \bigoplus_{n \geq 0} A^{\otimes n}$ on A is a graded ring under concatenation of tensors, with $T(A)_{kn} = A^{\otimes n}$ for any k. It is not graded commutative (usually).

<u>Ex</u> The polynomial ring $\mathbb{Z}[x_1, \dots, x_r]$ is a graded ring with $\mathbb{Z}[x_1, \dots, x_r]_{kn}$ the span of all degree n monomials. It is graded commutative if k is even.

<u>Ex</u> The same is true for the exterior algebra $\Lambda[x_1, \dots, x_r]$, except with graded commutativity for k odd.

Recall that the <u>cup product</u> of $\lambda \in H^p(X)$ and $\mu \in H^q(X)$ is the class of

$$C_n(X) \xrightarrow{\Delta^\#} C_n(X \times X) \xrightarrow{AW} C_p(X) \otimes C_q(X) \xrightarrow{\tilde{\lambda} \otimes \tilde{\mu}} \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\text{mult.}} \mathbb{Z}$$

where $p+q=n$.

<u>Thm</u> For any space X, $H^*(X)$ is a graded commutative ring under cup product, and the homomorphism induced by any map is a graded ring homomorphism.

<u>Ex</u> For degree reasons, the generator of $H^n(S^n)$ squares to 0, so

$$H^*(S^n) \cong \begin{cases} \Lambda[x], \ |x|=n & \text{if } n \text{ is odd} \\ \mathbb{Z}[x]/x^2, \ |x|=n & \text{if } n \text{ is even.} \end{cases}$$

<u>Lemma</u> Let $T : X \times Y \to Y \times X$ be the map $T_{X,Y}(x,y) = (y,x)$. The two composites in the following diagram are naturally chain homotopic:

$$\begin{tikzcd}
C_*(X \times Y) \arrow[r, "AW"] \arrow[d, "(T)_\#"'] & C_*(X) \otimes C_*(Y) & (-1)^{|\sigma_1||\sigma_2|}\sigma_2 \otimes \sigma_1 \\
C_*(Y \times X) \arrow[r, "AW"] & C_*(Y) \otimes C_*(X) \arrow[u, "\tau"'] & \sigma_1 \otimes \sigma_2 \arrow[u, mapsto]
\end{tikzcd}$$

<u>Proof</u> The two agree in degree 0, so it suffices by acyclic models to show that τ is a chain map, for which we have

$$\partial \tau(\sigma_1 \otimes \sigma_2) = (-1)^{|\sigma_1||\sigma_2|} \partial(\sigma_2 \otimes \sigma_1)$$

$$= (-1)^{|\sigma_1||\sigma_2|} \left[\partial\sigma_2 \otimes \sigma_1 + (-1)^{|\sigma_2|} \sigma_2 \otimes \partial\sigma_1 \right]$$

$$\tau \partial(\sigma_1 \otimes \sigma_2) = \tau\left(\partial\sigma_1 \otimes \sigma_2 + (-1)^{|\sigma_1|} \sigma_1 \otimes \partial\sigma_2 \right)$$

$$= (-1)^{(|\sigma_1|-1)|\sigma_2|} \sigma_2 \otimes \partial\sigma_1 + (-1)^{|\sigma_1| + |\sigma_1|(|\sigma_2|-1)} \partial\sigma_2 \otimes \sigma_1 .$$

$\square$

<u>Proof of theorem</u> Associativity follows from the explicit formula: both $\eta_1 \cup (\eta_2 \cup \eta_3)$ and $(\eta_1 \cup \eta_2) \cup \eta_3$ are equal to

$$\sigma \longmapsto \tilde{\pi}_1(\sigma|_{[e_0,\dots,e_p]})\,\tilde{\pi}_2(\sigma|_{[\sigma_p,\dots,\sigma_{p+q}]})\,\tilde{\pi}_3(\sigma|_{[\sigma_{p+q},\dots,\sigma_n]})$$

(or generalize acyclic models to $X \times Y \times Z$).

By inspection, the function

$$C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

$$\sum_i n_i \sigma_i \longmapsto \sum n_i$$

is a multiplicative unit, and the equation $f^*(\lambda \cup \mu) = f^*\lambda \cup f^*\mu$ follows either by inspection or from naturality of all maps involved. For graded commutativity, consider the diagram

$$\begin{tikzcd}
 & C_n(X\times X) \arrow[r, "AW"] \arrow[dd, "T_\#"] & C_p(X)\otimes C_q(X) \arrow[dd, "\tau"] \arrow[dr, "\tilde{\lambda}\otimes\tilde{\mu}"] & & \\
C_n(X) \arrow[ur, "\Delta\#"] \arrow[dr, "\Delta_\#"'] & & & \mathbb{Z}\otimes\mathbb{Z} \arrow[r, "\text{multiply}"] & \mathbb{Z} \\
 & C_n(X\times X) \arrow[r, "AW"] & C_p(X)\otimes C_q(X) \arrow[ur, "(-1)^{pq}\tilde{\mu}\otimes\tilde{\lambda}"'] & &
\end{tikzcd}$$

The top composite represents $\lambda\cup\mu$ and the bottom $(-1)^{pq}\mu\cup\lambda$. The triangles commute and the square up to chain homotopy by the lemma. Thus, the difference of the composites is of the form

$$\tilde{\lambda}\otimes\tilde{\mu}\circ(\partial\Gamma+\Gamma\partial)\circ\Delta_{\#}=\tilde{\lambda}\otimes\tilde{\mu}\circ\Gamma\circ\Delta_{\#}\circ\partial,$$

a coboundary (we use that $\tilde{\lambda}$ and $\tilde{\mu}$ are cocycles and that $\Delta_{\#}$ is a chain map.

$\square$