Last time

- Examples of graded rings
- Koszul signs
- $H^*(X)$ graded commutative

Since the cup product involves the Alexander-Whitney map, it is natural to look for a connection with the Künneth theorem.

Def The *tensor product* of the graded rings R and S is the graded ring with

$$(R \otimes S)_n = \bigoplus_{p+q=n} R_p \otimes S_q \quad \text{and}$$

$$(r_1 \otimes s_1)(r_2 \otimes s_2) = (-1)^{|s_1||r_2|} r_1 r_2 \otimes s_1 s_2 .$$

<u>Exercise</u> $R \otimes S$ is graded commutative if R and S are so.

<u>Ex</u> $\mathbb{Z}[x] \otimes \mathbb{Z}[y] \cong \mathbb{Z}[x,y]$ for $|x|$ and $|y|$ even

<u>Ex</u> $\Lambda[x] \otimes \Lambda[y] \cong \Lambda[x,y]$ for $|x|$ and $|y|$ odd

<u>Observation</u> If R is a graded commutative ring then multiplication $R \otimes R \xrightarrow{m} R$ is a map of graded rings. This amounts to the equation

$$m((r_1 \otimes r_2)(r_3 \otimes r_4)) = m(r_1 \otimes r_2)\, m(r_3 \otimes r_4).$$

The LHS is

$$m\left((-1)^{|r_2||r_3|} r_1 r_3 \otimes r_2 r_4\right) = (-1)^{|r_2||r_3|} r_1 r_3 r_2 r_4$$
$$= (-1)^{|r_2||r_3| + |r_3||r_2|} r_1 r_2 r_3 r_4$$
$$= m(r_1 \otimes r_2) m(r_3 \otimes r_4).$$

Thus, the composite ("cohomology cross product")

$$H^*(X) \otimes H^*(Y) \xrightarrow{\pi_1^* \otimes \pi_2^*} H^*(X \times Y) \otimes H^*(X \times Y) \xrightarrow{\smile} H^*(X \times Y)$$

is a map of graded rings.

<u>Thm</u> (Künneth) The cross product is an isomorphism provided $H^*(X)$ and $H^*(Y)$ are free Abelian and one is of finite type (i.e., finite rank in each degree).

__Lemma__ Let A and B be free Abelian groups. The canonical map

$$A^\vee \otimes B^\vee \longrightarrow (A \otimes B)^\vee$$
$$\lambda \otimes \mu \longmapsto (a \otimes b \longmapsto \lambda(a)\mu(b))$$

is an isomorphism if and only if A or B has finite rank.

__Proof__ Supposing $A \cong \mathbb{Z}^r$, we have

$$\begin{aligned} A^\vee \otimes B^\vee &\cong (\mathbb{Z}^r)^\vee \otimes B^\vee \\ &\cong \mathbb{Z}^r \otimes B^\vee \\ &\cong (B^\vee)^r \\ &\cong (B^r)^\vee \end{aligned}$$

$$\cong (\mathbb{Z}^r \otimes B)^\vee$$

$$\cong (A \otimes B)^\vee,$$

and one checks that the map in question is this isomorphism. For the converse, supposing (for simplicity) that $A = B = \bigoplus_{\mathbb{N}} \mathbb{Z}$, the functional

$$\delta(e_i \otimes e_j) = \delta_{ij}$$

is not in the image of $A^\vee \otimes B^\vee \longrightarrow (A \otimes B)^\vee$.

$\square$

Proof of theorem It is an exercise in unraveling the definitions to show that the diagram

$$
\begin{tikzcd}
H^*(X) \otimes H^*(Y) \arrow[rr, "\times"] \arrow[dr] \arrow[dd, "(1)"'] & & H^*(X \times Y) \arrow[dd, "(2)"] \\
& H^*(C^*(X) \otimes C^*(Y)) \arrow[dd] & \\
H_*(X)^\vee \otimes H_*(Y)^\vee \arrow[dd, "(3)"'] & & H_*(X \times Y)^\vee \\
& H^*((C_*(X) \otimes C_*(Y))^\vee) \arrow[dr] & \\
(H_*(X) \otimes H_*(Y))^\vee & & H_*(C_*(X) \otimes C_*(Y))^\vee \arrow[ll, "(5)", "\text{Künneth}^\vee"'] \arrow[uu, "AW^\vee", "(4)"']
\end{tikzcd}
$$

commutes. By freeness and Künneth, $H_*(X \times Y)$ is also free, so arrows (1) and (2) are isomorphisms by Künneth, and arrow (3) is by the lemma (we use finiteness). Arrow (4) is the dual of an isomorphism, and (5) is an isomorphism by freeness.

$\square$

Ex We have

$$H_n(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}^2 & n=1 \\ 0 & \text{otherwise}, \end{cases}$$

so homology can't distinguish this space from the torus. By the theorem, however, we have

$$H^*(T^2) \cong H^*(S^1) \otimes H^*(S^1)$$
$$\cong \Lambda[x] \otimes \Lambda[y]$$
$$\cong \Lambda[x,y];$$

in particular, $xy \neq 0$. On the other hand, the inclusion of $S^1 \vee S^1 \subseteq S^1 \vee S^1 \vee S^2$ is a retract, so the generators $x, y \in H^1(S^1 \vee S^1 \vee S^2)$ lie in the image of the ring map $H^*(S^1 \vee S^1) \xrightarrow{r^*} H^*(S^1 \vee S^1 \vee S^2)$. Thus,

$$xy = r^*(\tilde{x}) r^*(\tilde{y})$$
$$= r^*(\tilde{x}\tilde{y})$$
$$= 0$$

since $\tilde{x}\tilde{y} \in H^2(S^1 \vee S^1) = 0$.

Thus, $H^*(S^1 \vee S^1 \vee S^2)$ and $H^*(T^2)$ are non-isomorphic graded rings, so $S^1 \vee S^1 \vee S^2$ and T^2 are not homotopy equivalent.