Last time

- Tensor products of graded rings
- $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$ as rings
- $S^1 \vee S^1 \vee S^2 \not\simeq T^2$

More generally, we have the following calculation.

<u>Cor</u> Given $m_i > 0$ odd and $n_j > 0$ even, there there is an isomorphism of graded rings

$$H^*\left(\prod_{i=1}^{r} S^{m_i} \times \prod_{j=1}^{s} S^{n_j}\right) \cong \Lambda[x_1, \dots, x_r] \otimes \mathbb{Z}[y_1, \dots, y_s] / (y_1^2, \dots, y_s^2)$$

with $|x_i| = m_i$ and $|y_j| = n_j$.

In particular, we have

$$\operatorname{rk} H^k(T^n) = \binom{n}{k} = \binom{n}{n-k} = \operatorname{rk} H_{n-k}(T^n).$$

This symmetry is an instance of a much more general phenomenon, called <u>Poincaré duality</u>.

<u>Def</u> A <u>manifold</u> of dimension n (or n-manifold) is a Hausdorff space M locally homeomorphic to $\mathbb{R}^n$. A <u>local orientation</u> at $x \in M$ is a generator of

$$H_n(M, M \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$
$$\cong H_{n-1}(\mathbb{R}^n \setminus \{0\})$$
$$\cong H_{n-1}(S^{n-1})$$
$$\cong \mathbb{Z}.$$

We topologize the set of pairs

$$M_{\mathbb{Z}} = \{ (x, \alpha_x) \mid x \in M,\ \alpha_x \in H_n(M, M \setminus \{x\}) \}$$

via the following construction: given an open set $U \subseteq M$ and a class $\alpha \in H_n(M, M \setminus \overline{U})$

define

$$U_\alpha = \{ (x, \alpha|_{(M, M \setminus \{x\})}) : x \in U \}.$$

<u>Exercise</u> The collection $\{U_\alpha\}$ is a topological basis, and the map

$$\begin{aligned} M_{\mathbb{Z}} &\longrightarrow M \\ (x, \alpha_x) &\longmapsto x \end{aligned}$$

is a covering map.

Consequently, the subspace $\tilde{M} \subseteq M_{\mathbb{Z}}$ of pairs (x, α_x) with α_x a local orientation is a double cover.

<u>Prop/Def</u> The following are equivalent for an n-manifold M.

(1) $\tilde{M}$ is the trivial double cover.

(2) $\pi : \tilde{M} \to M$ admits a section.

(3) π admits a section over every compact subspace of M.

(4) For any compact $K \subseteq M$, there exists

$\alpha \in H_n(M, M \setminus K)$ restricting to a local orientation at every $x \in K$.

When these conditions hold, we say that M is <u>orientable</u>.

<u>Thm</u> (Poincaré duality) Let M be a compact, connected, orientable n-manifold. There is a canonical isomorphism

$$H^k(M) \cong H_{n-k}(M)$$

for every $k \in \mathbb{Z}$.

Rmk In particular, H_i and H^i are zero for $i > n$.

In order to prove this result, we require more knowledge of covering spaces.

Prop Let $p: E \to B$ be a covering map and γ a path in B with $\gamma(0) = b_0$. For every $e_0 \in p^{-1}(b_0)$, there is unique lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = e_0$:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"] \arrow[ur, dashed, "\exists! \tilde{\gamma}"] & B \end{tikzcd}$$

Proof Write $B = \bigcup_{i \in I} \mathcal{U}_i$, where each $\mathcal{U}_i$ is evenly covered by p. By compactness, there exist $0 = s_0 < s_1 < \cdots < s_{r-1} < s_r = 1$ such that

$\gamma([s_{k-1}, s_k]) \subseteq \mathcal{U}_{i_k}$ for some $i_k \in I$

and all $0 < k \leq r$ (take $s_{k+1} - s_k$ smaller than the Lebesgue number of $\{\gamma^{-1}(\mathcal{U}_i)\}_{i \in I}$). Given $e_0 \in p^{-1}(b_0)$, we will show that there is a unique lift $\tilde{\gamma}_k$ of $\gamma|_{[0, s_k]}$ starting at e_0 for every $0 \leq k \leq r$. The case $k=0$ is trivial, and the case $k=r$ is the desired result. Suppose there is such a unique lift $\tilde{\gamma}_{k-1}$. By uniqueness, any lift of $\gamma|_{[0,s_k]}$ agrees with $\tilde{\gamma}_{[0,s_{k-1}]}$, so it suffices by the pasting lemma to show the existence of a unique lift of $\gamma|_{[s_{k-1},s_k]}$ starting at $\tilde{\gamma}_{k-1}(s_{k-1})$. Since U_{i_k} is evenly covered by p, there is an open set $V \subseteq E$ such that $\tilde{\gamma}_{k-1}(s_{k-1}) \in V$ and $p|_V : V \xrightarrow{\cong} U_k$. Then

$(p|_V)^{-1} \circ \gamma|_{[s_{k-1},s_k]}$ is the unique lift. □

$$\begin{tikzcd}
 & V \arrow[r, hook] \arrow[dr, "\cong"] & \bigcup_{j\in J} V_j \arrow[d, "p"] \\
\{s_{k-1}\} \arrow[r, hook] \arrow[ur, "\tilde{\gamma}_{k-1}"] & {[s_{k-1},s_k]} \arrow[r, "\gamma"] \arrow[u, dashed] & U_{i_k}
\end{tikzcd}$$