Last time

- Cohomology rings of tori
- Manifolds and (local) orientations
- Poincaré duality
- Unique path lifting.

Corollary Let $p: E \to B$ be a covering map. If every loop in B lifts to a loop in E, then E is trivial.

Proof The hypothesis implies that every path component of E is a 1-fold cover. $\square$

Prop The following are equivalent for an n-manifold M.

(1) $\tilde{M}$ is the trivial double cover.

(2) $\pi : \tilde{M} \to M$ admits a section.

(3) π admits a section over every compact subspace of M.

(4) For any compact $K \subseteq M$, there exists $\alpha \in H_n(M, M \setminus K)$ restricting to a local orientation at every $x \in K$.

Lemma For compact K, $H_k(M, M \setminus K) = 0$ for $k > n$.

Proof of proposition The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious, and $(3) \Rightarrow (1)$ by the corollary, since the image of a loop is compact. Assuming (4) for K, the assignment

$$x \longmapsto (x, \alpha|_{(M, M \setminus \{x\})})$$

is a section over K, so $(4) \Rightarrow (3)$.

To show that $(3) \Rightarrow (4)$, we will show that, given a section $x \longmapsto (x, \alpha_x)$ of $M_{\mathbb{Z}}$ over K, there is a <u>unique</u> class $\alpha_K \in H_n(M, M \setminus K)$ restricting to α_x for $x \in K$.

We begin by noting that, if $K \subseteq \mathbb{R}^n \subseteq M$ is convex, then the remaining arrows in

$$\begin{tikzcd}
H_n(M, M\setminus K) \arrow[r] & H_n(M, M\setminus\{x\}) \\
H_n(\mathbb{R}^n, \mathbb{R}^n\setminus K) \arrow[r] \arrow[u, "\cong"] & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\}) \arrow[u, "\cong"']
\end{tikzcd}$$

are isomorphisms for $x \in K$, since $\mathbb{R}^n \setminus K$ deformation retracts onto $\partial \overline{B}_R(x)$ for R sufficiently large, so the claim holds in this case.

Next, if $K = K_1 \cup K_2$ with K_1 and K_2 compact convex subsets of a Euclidean neighborhood, we have the exact sequence

$$\cancelto{0}{H_{n+1}(M, M\setminus K_1\cap K_2)} \to H_n(M, M\setminus K) \qquad \exists!\ \alpha_K$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\alpha_{K_i} \qquad H_n(M, M\setminus K_1)\oplus H_n(M, M\setminus K_2) \qquad (\alpha_{K_1}, -\alpha_{K_2})$$

$$\downarrow \qquad\qquad \downarrow$$

$$\alpha_{K_1\cap K_2} \qquad H_n(M, M\setminus K_1\cap K_2).$$

Thus, the claim holds for a finite union of such.

If $K\subseteq \mathbb{R}^n\subseteq M$ is arbitrary, existence of α_K follows by restricting $\alpha_{\overline{B_R(0)}}$ for $K\subseteq B_R(0)$. For uniqueness, K may be covered by finitly many closed balls disjoint from the images of the simplices in ∂z, where $[z] = \alpha_K - \alpha_K'$, and we may apply the previous case to this union and the zero section.

Finally, an arbitrary K is a union of finitely many compact subsets of Euclidean neighbourhoods, and the same Mayer–Vietoris argument applies. $\square$

<u>Exercise</u> Imitate this argument to prove the lemma.

Cor A compact, connected n-manifold M is orientable iff $H_n(M) \cong \mathbb{Z}$.

Cor The following manifolds are orientable.

- S^n, $n \geq 0$
- Σ_g, $g \geq 0$
- $\mathbb{RP}^n$, n odd
- $\mathbb{CP}^n$, $n \geq 0$

The following manifolds are not orientable.

- $\mathbb{RP}^n$, n even
- The Klein bottle.

Def An orientation of M is a section of $\tilde{M} \to M$.

In the compact case, a choice of orientation is equivalent to a choice of generator for $H_n(M)$, called a fundamental class. With this class, we will construct the map relating $H^k(M)$ and $H_{n-k}(M)$.

Def The <u>cap product</u> is the homomorphism

$$H_k(X)\otimes H^\ell(X) \xrightarrow{\ \frown\ } H_{k-\ell}(X)$$

induced by the map

$$C_*(X)\otimes C^*(X) \xrightarrow{\Delta_\#\otimes \mathrm{id}} C_*(X\times X)\otimes C^*(X)$$

$$\xrightarrow{AW\otimes \mathrm{id}} C_*(X)\otimes C_*(X)\otimes C^*(X) \xrightarrow{\mathrm{id}\otimes \mathrm{ev}} C_*(X)\otimes \mathbb{Z} = C_*(X).$$

Explicitly, $\sigma\otimes\lambda \longmapsto \lambda(\sigma|_{[e_0,\dots,e_\ell]})\,\sigma|_{[e_\ell,\dots,e_k]}$.

<u>Remark</u> Here we view $C^*(X)$ as a chain complex concentrated in negative degrees.

Thm v2 If M is compact, connected, and orientable with fundamental class α_M, then the homomorphism

$$\begin{aligned} H^k(M) &\xrightarrow{D} H_{n-k}(M) \\ \lambda &\longmapsto \alpha_M \frown \lambda \end{aligned}$$

is an isomorphism.