Last time

- Sections and relative homology
- Fundamental classes
- Cap product
- PD: $\alpha_M \frown (-) : H^k(M) \xrightarrow{\cong} H_{n-k}(M)$

We prove Poincaré duality later. For now, we explore some of its consequences.

<u>Exercise</u> $f_*(\alpha) \frown \lambda = f_*(\alpha \frown f^*(\lambda))$

<u>Exercise*</u> $\mu(\alpha \frown \lambda) = (\lambda \smile \mu)(\alpha)$

Def Let A and B be Abelian groups. A homomorphism $A \otimes B \xrightarrow{\langle -,- \rangle} \mathbb{Z}$ is a **non-degenerate pairing** if the homomorphisms

$$\begin{array}{ccc} A \longrightarrow B^\vee & \text{and} & B \longrightarrow A^\vee \\ a \longmapsto \langle a, - \rangle & & b \longmapsto \langle -, b \rangle \end{array}$$

are isomorphisms.

Cor For M as above with free Abelian (co)homology, the pairing

$$H^k(M) \otimes H^{n-k}(M) \xrightarrow{-\cup-} H^n(M) \xrightarrow{\lambda \mapsto \lambda(\alpha_M)} \mathbb{Z}$$

is non-degenerate.

__Proof__ The composite

$$H^{n-k}(M) \longrightarrow H_{n-k}(M)^\vee \xrightarrow{D^\vee} H^k(M)^\vee$$

sends λ to the functional

$$\mu \longmapsto \lambda(\alpha_M \frown \mu) = \lambda \smile \mu(\alpha_M)$$

by the starred exercise. The first map is an isomorphism by Künneth and the second by PD. Since the cup product is graded commutative, the claim follows.

$\square$

__Cor__ The element $\lambda \in H^k(M)$ generates an infinite cyclic summand iff $\lambda \cup \mu$ generates $H^n(M)$ for some $\mu \in H^\ell(M)$.

__Proof__ The hypothesis is equivalent to the existence of a functional $\varphi : H^k(M) \to \mathbb{Z}$ with $\varphi(\lambda) = 1$, and every functional is realized by the cup product pairing by non-degeneracy. $\square$

Ex We show by induction that

$$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/x^{n+1}, \quad |x|=2.$$

For the base case, $\mathbb{CP}^1 \cong S^2$ (or $\mathbb{CP}^0 = pt$).
On the homework, you show that

$$H^k(\mathbb{CP}^n) \cong H_k(\mathbb{CP}^n)^\vee \cong \begin{cases} \mathbb{Z} & k \leq 2n \text{ even} \\ 0 & \text{otherwise}, \end{cases}$$

and the inclusion $\mathbb{CP}^{n-1} \subseteq \mathbb{CP}^n$ induces isomorphisms below degree $2n$. Letting $x \in H^2(\mathbb{CP}^n)$ denote a generator, x^i generates $H^{2i}(\mathbb{CP}^n)$ for $i<n$ by induction.

Hence $x^{n-1}(mx) = mx^n$ generates $H^{2n}(\mathbb{CP}^n)$, whence $m=1$, implying the claim.

<u>Ex</u> A similar argument gives another calculation of the ring structure of $H^*(T^2)$. Writing $x, y \in H^1(T^2)$ for the canonical generators, PD implies that $H^2(T^2)$ is generated by

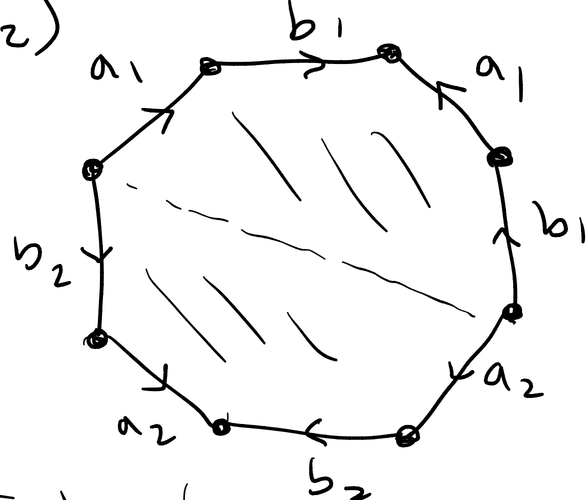
$$x(mx+ny) = nxy$$

since $2x^2=0 \Rightarrow x^2=0$, whence $n=1$.

It follows that $H^*(T^2) \cong \Lambda[x,y]$, as before. But we can do better: writing

$$H^1(\Sigma_2) \cong \mathbb{Z}\langle x_1, y_1, x_2, y_2 \rangle$$

with x_i dual to a_i and y_i dual to b_i, PD implies that $H^2(\Sigma_2)$ is generated by



$$x_1(my_1 + nx_2 + py_2),$$

since $x_1^2 = 0$. We claim that $x_1 x_2 = x_1 y_2 = 0$; indeed, the map $\Sigma_2 \to T^2 \vee T^2$ obtained by collapsing the dashed arc at right induces an isomorphism on H_1, hence on H^1. Since cohomology classes from different wedge factors multiply trivially (as in the example of $S^1 \vee S^1 \vee S^2$), the claim follows from consideration of the ring homomorphism

$$H^*(T^2 \vee T^2) \longrightarrow H^*(\Sigma_2).$$

As before, it follows that x, y, generates, So $x_1 y_1 = x_2 y_2$ by symmetry. Elaborating this argument, we conclude that

$$H^*(\Sigma_g) \cong \frac{\Lambda[x_i, y_i : 1 \le i \le g] \otimes \mathbb{Z}[z]}{x_i x_j = y_i y_j = 0, \quad x_i y_j = \delta_{ij} z}$$

We move toward a proof of Poincaré duality.

$$\begin{array}{ccc} H_n(M) & \longrightarrow & H_n(M, M \setminus \{x\}) \cong \mathbb{Z} \\ \parallel & & \\ \langle \alpha_M \rangle & & \end{array}$$

"fundamental class"

iso. for M cpt. conn. orientable

__PD__: $D: H^k(M) \xrightarrow{\alpha_M \frown -} H_{n-k}(M)$ iso.

Strategy

(1) $B_r(x) \subseteq \mathbb{R}^n$

(2) $U, V, + U \cap V \Rightarrow U \cup V$ (Mayer-Vietoris)

(3) $U_i \subseteq U_{i+1} \Rightarrow \bigcup U_i$

(4) Zorn's lemma

Problem Most of these are non-compact!