Last time

- Perfect pairings & PD
- $H^*(\mathbb{CP}^n)$ & $H^*(\Sigma_g)$ as rings

PD: $D: H^k(M) \xrightarrow{\alpha_M \frown -} H_{n-k}(M)$ iso.

Problem A "local-to-global" approach involves non-compact manifolds.

Exercise $H_n(M) = 0$ if M is non-compact.

So the theorem is certainly not true for non-compact manifolds.

Def A cochain $\lambda: C_*(X) \to \mathbb{Z}$ has **compact support** if there exists a compact subset $K \subseteq X$ such that $\lambda|_{C_*(X \setminus K)} \equiv 0$.

Exercise/Def The collection $C_c^*(X)$ of cochains with compact support is a subcomplex, whose cohomology is the **compactly supported cohomology** of X, denoted $H_c^*(X)$.

Rmk If X is compact, $C_c^*(X) = C^*(X)$.

Theorem v3 For M a connected orientable n-manifold, $H_c^k(M) \cong H_{n-k}(M)$.

Def A <u>directed set</u> is a partially ordered set in which every pair of elements has a common upper bound.

Ex The set of compact subspaces of a space X is directed under inclusion.

Def A <u>directed system</u> of Abelian groups is a directed set I, a group A_i for $i \in I$, and a homomorphism $\varphi_{ij} : A_i \longrightarrow A_j$ for every $i \leq j$, such that

(1) $\varphi_{ii} = \mathrm{id}_{A_i}$ for $i \in I$

(2) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for $i \leq j \leq k$.

The <u>direct limit</u> of this direct system is the set

$$\varinjlim_i A_i := \varinjlim A_i = \coprod_{i \in I} A_i \Big/ a \sim \varphi_{ij}(a),\ a \in A_i,\ i \leq j.$$

The direct limit becomes an Abelian group via the operation $[a] + [b] = [\varphi_{ik}(a) + \varphi_{jk}(b)]$ for $a \in A_i$, $b \in A_j$ and k a common upper bound for i and j.

<u>Ex</u> Rephrasing an earlier result from the course, $H_n(X) \cong \varinjlim H_n(U_i)$ for $X = \bigcup U_i$ an exhaustion by open sets.

__Lemma__ There is a canonical isomorphism

$$\varinjlim H^n(X, X \setminus K) \xrightarrow{\cong} H^n_c(X)$$

__Lemma__ Let $A_0 \subseteq A$ be a subgroup of an Abelian group. The assignment

$$\begin{aligned} (A/A_0)^\vee &\longrightarrow A^\vee \\ \lambda &\longmapsto (a \mapsto \lambda([a])) \end{aligned} \qquad \begin{array}{c} A \\ \downarrow \\ A/A_0 \\ \downarrow \lambda \\ \mathbb{Z} \end{array}$$

is an isomorphism onto $\{\mu \in A^\vee : \mu|_{A_0} = 0\}$.

__Proof__ If $\mu|_{A_0} = 0$, then the universal property of the quotient gives the desired filler in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d] & \mathbb{Z} \\ A/A_0 \arrow[ur, dashed, "\exists! \bar{\mu}"'] & \end{tikzcd},$$

which is surjectivity. Conversely, if $\mu = 0$ in the above diagram, then $\bar{\mu} = 0$, which is injectivity. $\square$

<u>Proof of previous lemma</u> For fixed K, we have

$$C^*(X, X \setminus K) = C_*(X, X \setminus K)^\vee$$
$$= \left(C_*(X)/C_*(X \setminus K)\right)^\vee$$

$$\cong \{\lambda \in C^*(X) \mid \lambda|_{C_*(X \setminus K)} = 0\}$$

$$\subseteq C_c^*(X).$$

Varying K, we obtain the well-defined homomorphism

$$\varinjlim H^*(X, X \setminus K) \longrightarrow H_c^*(X).$$

Surjectivity is essentially immediate, and, if $\partial\mu = \lambda$ with $\lambda|_{X \setminus K} = 0$ and $\mu|_{X \setminus L} = 0$, then $\lambda|_{X \setminus K \cup L} = \mu|_{X \setminus K \cup L} = 0$, so $[\lambda] = 0$ in $H^*(X, X \setminus K \cup L)$, hence in the direct limit. $\square$

Lemma Suppose given directed systems $\{A_i\}_{i\in I}$ and $\{A_i'\}_{i\in I}$ and homomorphisms $\psi_i : A_i \to A_i'$ such that the diagram

$$\begin{tikzcd} A_i \arrow[r, "\varphi_{ij}"] \arrow[d, "\psi_i"'] & A_j \arrow[d, "\psi_j"] \\ A_i' \arrow[r, "\varphi_{ij}'"] & A_j' \end{tikzcd}$$

commutes for every $i,j \in I$. Denoting the induced homomorphism by $\psi : \varinjlim A_i \to \varinjlim A_i'$, there are canonical isomorphisms

$$\varinjlim \ker\psi_i \xrightarrow{\cong} \ker\psi \qquad \varinjlim \operatorname{im}\psi_i \xrightarrow{\cong} \operatorname{im}\psi .$$

More pithily, we say "direct limits preserve kernels and images".

<u>Proof</u> We prove the first claim; the second is an exercise. An element in $\ker\psi$ is represented by $a \in A_i$. Since $\psi([a]) = 0$, we have

$$0 = \varphi_{ij}\psi_i(a) = \psi_j\varphi_{ij}(a)$$

for some $j \geq i$, so $[\varphi_{ij}(a)] \in \ker\psi_j$, which is surjectivity. Injectivity is similar. $\square$

<u>Cor</u> Direct limits preserve exactness.