Last time

- Compactly supported cohomology
- Direct limits

Corollary Suppose that $X = U \cup V$ with U and V open. There is a long exact sequence of the form

$$\cdots \to H^n_c(U) \oplus H^n_c(V) \to H^n_c(X) \to H^{n+1}_c(U \cap V) \to \cdots$$

Proof Given compact $K \subseteq U$ and $L \subseteq V$, there is the exact sequence

$$\cdots \to H^n(X, X\setminus K) \oplus H^n(X, X\setminus L) \to H^n(X, X\setminus K\cup L) \to H^{n+1}(X, X\setminus K\cap L) \to \cdots$$

Passing to direct limits over the directed set of such pairs (K,L) produces an exact sequence by the corollary, whose terms we identify.

(1) By excision $H^n(X, X\setminus K) \cong H^n(U, U\setminus K)$ and $H^n(X, X\setminus L) \cong H^n(V, V\setminus L)$, and

$$\varinjlim_{(K,L)} H^n(U, U\setminus K) \oplus H^n(V, V\setminus L)$$

$$\cong$$

$$\varinjlim_K H^n(U, U\setminus K) \oplus \varinjlim_L H^n(V, V\setminus L) = H^n_c(U) \oplus H^n_c(V)$$

(2) $\varinjlim_{(K,L)} H^n(X, X \setminus K \cup L) \cong \varinjlim_A H^n(X, X \setminus A) = H^n_c(X)$,

since compact $A \subseteq X$ is contained in the union of compact $A \cap (X \setminus V) \subseteq U$ and $A \cap (X \setminus U) \subseteq V$.

(3) $\varinjlim H^n(X, X \setminus K \cap L) \cong \varinjlim H^n(U \cap V, U \cap V \setminus K \cap L)$

$$= H^n_c(U \cap V),$$

similarly. $\square$

We now construct the duality map

$$D : H^k_c(M) \longrightarrow H_{n-k}(M).$$

To begin, we note that there is a

relative version of the cap product, given by the same formula:

$$\begin{tikzcd}
C_k(A) \otimes C^\ell(X,A) \arrow[d, hook] \arrow[drr, "0"] & & \\
C_k(X) \otimes C^\ell(X,A) \arrow[r] \arrow[d, two heads] & C_k(X) \otimes C^\ell(X) \arrow[r, "\frown"] & C_{k-\ell}(X) \\
C_k(X,A) \otimes C^\ell(X,A) \arrow[urr, dashed, "\exists!"'] & &
\end{tikzcd}$$

Construction

(1) Choose a section of $\tilde{M}$, gives rise to unique classes $\alpha_K \in H_n(M, M\setminus K)$.

(2) For $i : K \subseteq L$, the diagram

$$\begin{tikzcd}
H^k(M, M\setminus K) \arrow[dr, "\alpha_K \cap -"] \arrow[d, "i^*"'] & \\
H^k(M, M\setminus L) \arrow[r, "\alpha_L \cap -"'] & H_{n-k}(M)
\end{tikzcd}$$

commutes, since $i_* \alpha_L = \alpha_K$ by uniqueness, so

$$\alpha_L \cap i^* \lambda = i_* \alpha_L \cap \lambda = \alpha_K \cap \lambda .$$

(3) These homomorphisms induce

$$D: H^k_c(M) \cong \varinjlim H^k(M, M \setminus K) \dashrightarrow H_{n-k}(M)$$

<u>Thm</u> For M connected and oriented, the map

$$D: H^k_c(M) \longrightarrow H_{n-k}(M)$$

is an isomorphism.

<u>Lemma</u> | Suppose that $M = U \cup V$ with U and V open. The following diagram commutes up to sign:

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U) \oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^k_c(M) \arrow[r] \arrow[d, "D_M"] & H^{k+1}_c(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r, "\delta"] & H_{n-k-1}(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Remark on signs Different sign conventions are used for the cap product. With Hatcher's convention, it is only a chain map up to sign, which is good enough. Strict commutativity can be achieved by setting $\sigma \cap \lambda = (-1)^{k\ell} \lambda(\sigma|_{[e_0, \ldots, e_\ell]}) \sigma|_{[e_\ell, \ldots, e_k]}$.

We defer the proof of this lemma.

Corollary If PD holds for U, V, and $U \cap V$, then it holds for $U \cup V$.

Lemma 2 If PD holds for U_i and $U_i \subseteq U_{i+1}$, then it holds for $\bigcup_I U_i$.

Proof Since a compact subspace of U_i is also a compact subspace of U_{i+1}, we have the commuting diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U_i) \arrow[r, dashed] \arrow[d, "D_{U_i}", "\cong"'] & H^k_c(U_{i+1}) \arrow[r] \arrow[d, "D_{U_{i+1}}", "\cong"'] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U_i) \arrow[r] & H_{n-k}(U_{i+1}) \arrow[r] & \cdots
\end{tikzcd}$$

Forming direct limits, the claim follows.

$\square$

<u>Lemma 3</u> If PD holds for an arbitrary open subspace of $\mathbb{R}^n$, then it holds for M.

<u>Proof</u> If M admits a countable cover by open sets $U_i \cong \mathbb{R}^n$, apply Lemma 2 to the open exhaustion $\{\bigcup_{i \le j} U_i\}_{j \ge 0}$ to reduce to a finite union of Euclidean neighborhoods, then induct on the cardinality of the union using the corollary. In the uncountable case, use Zorn's lemma.

$\square$

<u>Lemma 4</u> If PD holds for $\mathbb{R}^n$, then it holds for an arbitrary open subspace thereof.

Proof By assumption and homeomorphism invariance, PD holds for any open ball, hence for any bounded convex open subspace of $\mathbb{R}^n$. An arbitrary open may be covered by open balls, and any finite intersection of these is bounded, convex, and open, so the argument of Lemma 3 applies. $\square$