Last time

- Mayer-Vietoris for H_c^*
- $D_M : H_c^k(M) \longrightarrow H_{n-k}(M)$
- Reduction to $\mathbb{R}^n$.

The next result is the one calculation underpinning Poincaré duality.

Lemma 5 The relative cap product

$$H^n(\Delta^n, \partial\Delta^n) \xrightarrow{[\mathrm{id}_{\Delta^n}] \cap -} H_0(\Delta^n)$$

is an isomorphism.

Proof Since $H_n(\Delta^n, \partial\Delta^n) = \mathbb{Z}\langle [\mathrm{id}_{\Delta^n}] \rangle$, we have $H^n(\Delta^n, \partial\Delta^n) = H_n(\Delta^n, \partial\Delta^n)^\vee = \mathbb{Z}\langle [\lambda] \rangle$, where $\lambda(k \cdot \mathrm{id}_{\Delta^n}) = k$. From the explicit formula,

$$\begin{aligned} \mathrm{id}_{\Delta^n} \frown \lambda &= \lambda(\mathrm{id}_{\Delta^n} \circ [e_0, \dots, e_n]) \cdot [e_n] \\ &= \lambda(\mathrm{id}_{\Delta^n}) \cdot [e_n] \\ &= [e_n], \end{aligned}$$

whose homology class generates $H_0(\Delta^n)$. $\square$

Proof of thm WLOG $M = \mathbb{R}^n$. Let Δ_N denote the n-simplex with barycenter 0 and diameter $N \in \mathbb{N}$, and consider the commuting diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \Delta_N) \arrow[rr] \arrow[dd, "D_{\Delta_N}"'] & & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \Delta_{N+1}) \arrow[r] \arrow[dd, "D_{\Delta_{N+1}}"] & \cdots \\
& & H^n(\Delta_N, \partial \Delta_N) \arrow[ul, "\cong"'] \arrow[dd, "{[\mathrm{id}_{\Delta^n}] \frown -}"] & & \\
& H_0(\mathbb{R}^n) \arrow[rr, equal] & & H_0(\mathbb{R}^n) & \\
& & H_0(\Delta_N) \arrow[ul, "\cong"'] & &
\end{tikzcd}
$$

By Lemma 5, each of the vertical arrows is an isomorphism, so the horizontal arrows are as well, and it follows that the induced homomorphism from $H^n_c(\mathbb{R}^n)$ is an isomorphism. $\square$

We have one loose end.

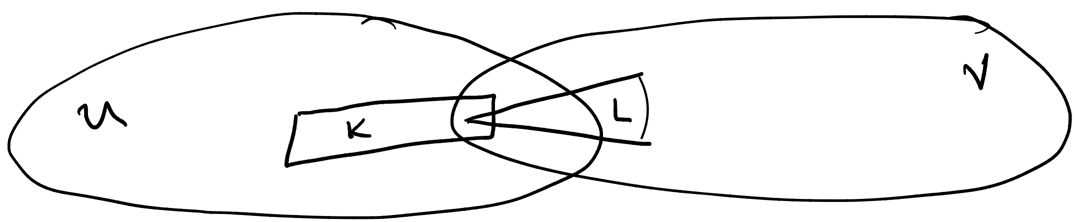
<u>Proof of Lemma 1</u> We wish to show that the diagram

$$\begin{tikzcd} H^k_c(M) \arrow[r] \arrow[d, "D_M"] & H^{k+1}_c(U \cap V) \arrow[d, "D_{U\cap V}"] \\ H_{n-k}(M) \arrow[r, "\delta"] & H_{n-k-1}(U \cap V) \end{tikzcd}$$

commutes. This diagram arises, forming a direct limit, from diagrams of the form

$$\begin{tikzcd}
H^k(M, M\setminus K\cup L) \arrow[r, "\delta"] \arrow[dd, "\alpha_{K\cup L}\frown -"'] & H^{k+1}(M, M\setminus K\cap L) \arrow[d, "\cong"] \\
& H^{k+1}(U\cap V, U\cap V\setminus K\cap L) \arrow[d, "\alpha_{K\cap L}\frown -"] \\
H_{n-k}(M) \arrow[r, "\delta"] & H_{n-k-1}(U\cap V)
\end{tikzcd}$$

for $K\subseteq U$ and $L\subseteq V$ compact. By subdivision, since $M = U\cup V = (U\setminus L)\cup(U\cap V)\cup(V\setminus K)$, we have



$$\alpha_{K\cup L} = [c_{U\setminus L} + c_{U\cap V} + c_{V\setminus K}]$$

with $c_{U\setminus L} \in C_n(U\setminus L, U\setminus K\cup L)$ and so on.

Then we have

$$\alpha_K = [c_{U\setminus L} + c_{U\cap V}]$$

$$\alpha_L = [c_{U\cap V} + c_{V\setminus K}]$$

$$\alpha_{K\cap L} = [c_{U\cap V}].$$

We calculate the value of the clockwise composite on $\lambda \in H^k(M, M\setminus K\cup L)$ as follows.

First, write $\eta = [\tilde{\eta}_{M\setminus K} + \tilde{\eta}_{M\setminus L}]$ with $\tilde{\eta}_{M\setminus K} \in H^K(M, M\setminus K)$ (resp. L). Then we have

$$\delta\eta = [\partial\tilde{\eta}_{M\setminus K}] = [\partial\tilde{\eta}_{M\setminus L}] \in H^{k+1}(M, M\setminus K\cap L),$$

so the end result is

$$[c_{U\cap V} \frown \partial\tilde{\eta}_{M\setminus K}] = [\partial c_{U\cap V} \frown \tilde{\eta}_{M\setminus K}],$$

since the difference of these chains is $\pm\,\partial(c_{U\cap V} \frown \tilde{\eta}_{M\setminus K})$, a boundary in $C_*(U\cap V)$.

To calculate the counterclockwise composite, we write

$$\delta(\alpha_{KUL} \wedge \lambda) = \delta\Big[\underbrace{c_{UIL} \wedge \tilde{\lambda}}_{\hat{C}_*(U)} + \underbrace{c_{UnV} \wedge \tilde{\lambda} + c_{VIK} \wedge \tilde{\lambda}}_{\hat{C}_*(V)}\Big]$$

$$= \Big[\partial(c_{UIL} \wedge \tilde{\lambda})\Big]$$

$$= \pm\Big[\partial c_{UIL} \wedge \tilde{\lambda} + (-1)^k c_{UIL} \wedge \cancelto{0}{\partial \tilde{\lambda}}\Big]$$

$$= \pm\Big[\partial c_{UIL} \wedge \tilde{\lambda}_{MIK} + \cancelto{0}{\partial c_{UIL} \wedge \tilde{\lambda}_{MIL}})\Big].$$

Thus, it suffices to show that

$$\partial c_{u \cap v} \frown \widetilde{\pi}_{M \setminus K} = \pm\, \partial c_{u \setminus L} \frown \widetilde{\pi}_{M \setminus K}.$$

For this, recall that $[c_{u \setminus L} + c_{u \cap v}] = \alpha_K$

so $\partial(c_{u \setminus L} + c_{u \cap v}) \in C_k(M \setminus K)$.

$\square$