Last time

- Proof of PD for $\mathbb{R}^n$
- Proof of Lemma 1

Q What about non-orientable manifolds?

A Repeat everything with coefficients in $\mathbb{F}_2$

As before, we have $H_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$, so there is a unique "local orientation" at every point, the analogue of $\widehat{M}$ is just $M$, and precisely the same argument proves the following.

<u>Thm</u> Let $M$ be a connected $n$-manifold. The duality map

$$D_M : H^k_c(M; \mathbb{F}_2) \longrightarrow H_{n-k}(M; \mathbb{F}_2)$$

is an isomorphism.

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To derive consequences for the cup product, we require a small digression.

Observation If $R$ is a commutative ring, then $C_*(X;R)$ is a chain complex of $R$-modules, so $H_*(X;R)$ is a graded $R$-module, and the whole theory of homology is $R$-linear. Since

$$C^*(X;R) = \mathrm{Hom}_{\mathbb{Z}}(C_*(X), R) \cong \mathrm{Hom}_R(C_*(X;R), R),$$

the same goes for cohomology. The cup product is $R$-bilinear, so $H^*(X;R)$ is a graded commutative $R$-algebra. The same arguments as before establish the universal coefficient and Künneth isomorphisms:

(1) $H_*(X;R) \otimes_R H_*(Y;R) \cong H_*(X \times Y;R)$

if $H_*(X;R)$ is free over $R$ in each degree

(2) $H^*(X;R) \cong H_*(X;R)^\vee$ ($R$-linear dual)

with the same assumption

(3) $H^*(X;R) \cong H_*(X;R)$ if $H_*(X;R)$ is also of finite type (as an $R$-module)

(4) $H^*(X;R) \otimes_R H^*(Y;R) \cong H^*(X \times Y;R)$ as graded commutative $R$-algebras if both are free of finite type.

Cor For $0 < n \leq \infty$ there is an isomorphism of graded $\mathbb{F}_2$-algebras

$$H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/x^{n+1}, \quad |x| = 1.$$

Proof From before, we have that

$$H^k(\mathbb{R}P^n; \mathbb{F}_2) \cong H_k(\mathbb{R}P^n; \mathbb{F}_2)^\vee \cong \mathbb{F}_2^\vee \cong \mathbb{F}_2.$$

Writing $x \in H^1(\mathbb{R}P^n; \mathbb{F}_2)$ for the generator, it suffices to show that $x^k \neq 0$ for $0 \leq k \leq n$. For $n < \infty$, Poincaré duality applies, as for $\mathbb{C}P^n$. For $n = \infty$, the inclusion

$\mathbb{RP}^k \subseteq \mathbb{RP}^\infty$ induces an isomorphism on $\mathbb{F}_2$-cohomology in degrees $\leq k$. Since the induced map is a ring homomorphism and $0 \neq x^k \in H^k(\mathbb{RP}^k; \mathbb{F}_2)$ by the previous case, the claim follows. $\square$

<u>Cor</u> There is an isomorphism of graded $\mathbb{F}_2$-algebras

$$H^*(\mathbb{RP}^n \times \mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x,y]/(x^{n+1}, y^{n+1}).$$

We give an application of this result to a classical problem in algebra.

**Def** A <u>division algebra</u> over the field $k$ is a unital $k$-algebra in which every nonzero element is a unit.

**Thm** (Hopf) A division algebra over $\mathbb{R}$ has dimension a power of 2.

**Rmk** In fact, the options are 1, 2, and 4 (Frobenius).

**Lemma** Fix a prime $p$ and write

$$a = \sum_{i \geq 0} a_i p^i, \quad b = \sum_{i \geq 0} b_i p^i \text{ with } 0 \leq a_i, b_i < p.$$

Then

$$\binom{b}{a} \equiv \prod_{i \geq 0} \binom{b_i}{a_i} \bmod p.$$

**Proof** In $\mathbb{F}_p[x]$,

$$(1+x)^b = \prod_{i\geq 0}(1+x)^{b_i p^i}$$
$$= \prod_{i\geq 0}(1+x^{p^i})^{b_i}.$$

The coefficient of $x^a$ on the LHS is $\binom{b}{a}$, and on the RHS it is $\prod_{i\geq 0}\binom{b_i}{a_i}$.

$\square$

**Corollary** If $\binom{n}{k}$ is even for every $0<k<n$, then $n$ is a power of 2.

<u>Proof of thm</u> Given such a structure on $\mathbb{R}^n$, define

$$S^{n-1} \times S^{n-1} \xrightarrow{\bar{f}} S^{n-1}$$
$$(v, w) \longmapsto \frac{vw}{|vw|}$$

Since $(-v)w = -vw = v(-w)$ by bilinearity, the induced map $f : \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^{n-1}$ is well-defined. We claim the map

$$f^* : \mathbb{F}_2[x]/x^n \longrightarrow \mathbb{F}_2[y,z]/(y^n, z^n)$$

satisfies $f^* x = y + z$. Assuming so, we have

$$0 = f^* x^n = (y+z)^n = \sum_{k=0}^{n} \binom{n}{k} y^k z^{n-k},$$

from which it follows that $\binom{n}{k}$ is even for $0<k<n$, and the corollary applies.

To prove the claim, since (coefficients mod 2)

$$H^1(\mathbb{R}P^{n-1}\times\mathbb{R}P^{n-1}) \cong H_1(\mathbb{R}P^{n-1}\times\mathbb{R}P^{n-1})^\vee$$
$$\cong H_1(\mathbb{R}P^{n-1})^\vee \oplus H_1(\mathbb{R}P^{n-1})^\vee,$$

it suffices to show that

$$f_*: H_1(\mathbb{R}P^{n-1})\oplus H_1(\mathbb{R}P^{n-1}) \longrightarrow H_1(\mathbb{R}P^{n-1})$$

is nonzero the first (by symmetry) summand.

But, writing $e$ for the division algebra unit and $u=\frac{e}{|e|}$, we have

$vu = v\frac{e}{|e|} = \frac{1}{|e|}(ve) = \frac{1}{|e|}v$ by bilinearity, so the composite

$$S^{n-1} \xrightarrow{(\mathrm{id}, u)} S^{n-1} \times S^{n-1} \xrightarrow{\bar{f}} S^{n-1}$$

is the identity. Descending to the quotient, it follows that the composite

$$H_1(\mathbb{RP}^{n-1}) \subseteq H_1(\mathbb{RP}^{n-1}) \oplus H_1(\mathbb{RP}^{n-1}) \longrightarrow H_1(\mathbb{RP}^{n-1})$$

is the identity, as desired. $\square$