Last time

- PD over $\mathbb{F}_2$ w/o orientation
- $H^*(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2[x]/x^{n+1}$
- Division algebras

Recall the generalized Jordan curve theorem, which is the isomorphism

$$\tilde{H}_i(S^n \setminus \mathrm{im}(f)) \cong \begin{cases} \mathbb{Z}, & i = n-r-1 \\ 0, & \text{otherwise} \end{cases}$$

for an embedding $f: S^r \hookrightarrow S^n$. In other words, we have

$$\tilde{H}_i(S^n \setminus \mathrm{im}(f)) \cong \tilde{H}^{n-i-1}(\mathrm{im}(f)).$$

The resemblance to Poincaré duality is no coincidence.

<u>Def</u> A space X is called <u>locally contractible</u> if, for every $x \in X$ and open $x \in U$, there is an open $x \in V \subseteq U$ such that the inclusion $V \subseteq U$ is nullhomotopic.

<u>Warning</u> There is disagreement about this terminology.

Thm (Alexander duality) If K is a compact locally contractible, non-empty, proper subspace of S^n, then
$$\tilde{H}_i(S^n \setminus K) \cong \tilde{H}^{n-i-1}(K).$$

For the proof, we need a technical result, which we return to later, given time.

Retract theorem A compact subspace of $\mathbb{R}^n$ is a retract of a neighborhood iff it is locally contractible.

Lemma If $K \subseteq \mathbb{R}^n$ is locally contractible, then the natural homomorphism

$$\varinjlim H^*(U) \to H^*(K)$$

is an isomorphism, where the direct limit is over the set of open neighborhoods of K.

Proof By the theorem, there is an open $K \subseteq U_0$ and a retraction $r : U_0 \to K$, so the composition

$$\begin{tikzcd} \varinjlim H^*(U) \arrow[r] & H^*(K) \arrow[dl, "r^*"] \\ H^*(U_0) \arrow[u] & \end{tikzcd}$$

is the identity, implying surjectivity. For injectivity, suppose that $\alpha \in H^*(U)$ maps to 0 in $H^*(K)$. We may assume that $U \subseteq U_0$, and consider the straight-line homotopy

$$H : \mathcal{U} \times [0,1] \to \mathbb{R}^n$$

from the inclusion $\mathcal{U} \subseteq \mathbb{R}^n$ to $r|_{\mathcal{U}}$. The open set $H^{-1}\mathcal{U}$ contains the slice $K \times [0,1]$, hence the tube $V \times [0,1]$ for some given $K \subseteq V \subseteq \mathcal{U}$ by compactness of $[0,1]$. Thus, $H|_{V \times [0,1]}$ is a homotopy from $V \subseteq \mathcal{U}$ to the composite $V \xrightarrow{r} K \subseteq \mathcal{U}$, implying that the diagram

$$\begin{tikzcd} H^*(U) \arrow[rr] \arrow[dr] & & H^*(V) \\ & H^*(K) \arrow[ur, "(r|_V)^*"'] & \end{tikzcd}$$

commutes. Since α maps to 0 in $H^*(K)$, it maps to 0 in $H^*(V)$, hence in the direct limit, implying injectivity.

$\square$

<u>Proof of AD</u> Consider the composition

$$
\begin{aligned}
H_i(S^n \setminus K) &\xleftarrow{D} H_c^{n-i}(S^n \setminus K) \\
&\leftarrow \varinjlim_{U} H^{n-i}(S^n \setminus K, U \setminus K) \\
&\leftarrow \varinjlim_{U} H^{n-i}(S^n, U) \\
&\xleftarrow{\delta} \varinjlim_{U} \tilde{H}^{n-i-1}(U) \\
&\longrightarrow \tilde{H}^{n-i-1}(K),
\end{aligned}
$$

where the limit is over open subsets $K \subseteq U \subseteq S^n$.

The first is an isomorphism by PD. The

second is an isomorphism in light of the bijection

$$\left\{\begin{matrix}\text{open} \\ K \subseteq U \subseteq S^n\end{matrix}\right\} \underset{S^n \setminus (-)}{\overset{S^n \setminus (-)}{\rightleftarrows}} \left\{\begin{matrix}\text{compact} \\ L \subseteq S^n \setminus K\end{matrix}\right\}.$$

The third is an isomorphism by excision.
The fifth is an isomorphism by the lemma.

For $i \neq 0$, the fourth is an isomorphism by exactness:

$$\cancelto{0}{\tilde{H}^{n-i-1}(S^n)} \longrightarrow \tilde{H}^{n-i-1}(U) \xrightarrow{\delta} H^{n-i}(S^n, U) \longrightarrow \cancelto{0}{\tilde{H}^{n-i}(S^n)}$$

For $i=0$, consider the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{H}_0(S^n \setminus K) \arrow[r] & H_0(S^n \setminus K) \arrow[r] & H_0(S^n) \arrow[r] & 0 \\
0 \arrow[r] & \ker \varphi \arrow[r] \arrow[u, dashed] & H^n_c(S^n \setminus K) \arrow[r, "\varphi"] \arrow[u, "D"'] & H^n(S^n) \arrow[r] \arrow[u, "D"'] & 0 \\
0 \arrow[r] & \ker \psi \arrow[r] \arrow[u, dashed] & \varinjlim H^n(S^n \setminus K, U \setminus K) \arrow[r, "\psi"] \arrow[u] & H^n(S^n) \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & \varinjlim H^{n-1}(U) \arrow[r, "\delta"] \arrow[u, dashed] & \varinjlim H^n(S^n, U) \arrow[r] \arrow[u] & H^n(S^n) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Since the middle and righthand vertical maps are isomorphisms, so are the lefthand vertical isomorphisms. $\square$