Last time

- Alexander duality $\tilde{H}_i(S^n \setminus K) \cong \tilde{H}^{n-i-1}(K)$

Corollary If $K \subseteq \mathbb{R}^n$ is compact and locally contractible, then $H_k(K)$ is trivial for $k \geq n$ and free Abelian for $k = n-1$.

Proof Applying AD to $K \subseteq \mathbb{R}^n \subseteq S^n$, note that $n-i-1 = k \geq n$ iff $i \leq -1$ and $n-i-1 = k = n-1$ iff $i=0$. Since $\tilde{H}_{<0}(S^n \setminus K) = 0$ and $\tilde{H}_0(S^n \setminus K)$ is free, the claim follows from the UCT. $\square$

To exploit this, we require a little more information on the homology of manifolds.

<u>Thm</u> Let M be a connected n-manifold.

(1) $H_k(M;A) = 0$ for $k > n$ and any A.

(2) $H_n(M;A) \cong \begin{cases} A & M \text{ compact, orientable} \\ \{a \in A : 2a = 0\} & M \text{ compact, nonorientable} \\ 0 & \text{otherwise.} \end{cases}$

(3) $H_{n-1}(M)$ is torsion-free unless M is compact and non-orientable, in which case it contains an element of order 2.

<u>Rmk</u> Using the general UCT, one can show that the torsion subgroup of $H_{n-1}(M)$ is exactly $\mathbb{Z}/2\mathbb{Z}$ in the last case.

Proof The first two are essentially known except for the non-compact, nonorientable case, for which we refer to Prop. 3.29 in Hatcher, which is a familiar argument.

For (3), the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

gives rise to the long exact sequence

$$H_n(M) \to H_n(M;\mathbb{Z}/m\mathbb{Z}) \to H_{n-1}(M) \xrightarrow{m} H_{n-1}(M).$$

If M is non-compact, the first two terms vanish, so multiplication by m is injective on $H_{n-1}(M)$ for every m. If M is compact and orientable, it is easy to see by considering $\widetilde{M}$ (for example) that the first map is surjective for every m, implying the same conclusion. If M is compact non-orientable, we have the sequence

$$0 \to \underset{\substack{\rotatebox{90}{$\cong$} \\ \mathbb{F}_2}}{H_n(M;\mathbb{F}_2)} \to H_{n-1}(M).$$

$\square$

Corollary Let M be a compact, connected n-manifold. If M embeds in $\mathbb{R}^{n+1}$, then M is orientable, and $\mathbb{R}^{n+1} \setminus M$ has two path components.

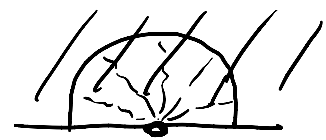
In particular, $\mathbb{RP}^n$ does not embed in $\mathbb{R}^{n+1}$.

Rephrasing the second claim, a compact connected n-manifold embedded in $\mathbb{R}^{n+1}$ "encloses space" or <u>bounds</u>. One can ask this question without reference to an embedding.

<u>Def</u> A <u>n-manifold with boundary</u> is a Hausdorff space M locally homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n \geq 0\}$.

If $x \in M$ has an open neighborhood $U \cong \mathbb{R}^n$, then $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ as before by excision. If $U \cong \mathbb{R}^n_+$ with x sent to the subspace $x_n = 0$, then

$$H_n(M, M\setminus\{x\}) \cong H_n(\mathbb{R}^n_+, \mathbb{R}^n_+\setminus\{0\})$$
$$\cong H_n(\mathbb{R}^n_+, S^{n-1}_+)$$
$$\cong 0$$



<u>Def</u> The <u>boundary</u> of the n-manifold with boundary M is

$$\partial M = \{x \in M \mid H_n(M, M\setminus\{x\}) = 0\}.$$

The <u>interior</u> is $\mathring{M} = M \setminus \partial M$.

Note that $\mathring{M}$ is an n-manifold, ∂M an $(n-1)$-manifold, and $\partial M \subseteq M$ is closed.

<u>Ex</u> D^n is a manifold with boundary S^{n-1}.

<u>Ex</u> $M \times [0,1]$ is a manifold with boundary $M \amalg M$.

<u>Q</u> Which manifolds are boundaries?

As before, this question is related to duality.

<u>Def</u> We say a manifold with boundary is <u>orientable</u> if its interior is so.

Lemma If M is a compact, orientable n-manifold with boundary, then

$$H_n(M, \partial M) \cong \mathbb{Z}.$$

So M has a fundamental class $\alpha_M \in H_n(M, \partial M)$.

Thm (Lefschetz duality) Let M be a compact, connected, oriented n-manifold with boundary. The homomorphism

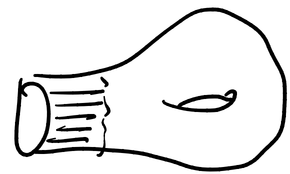
$$\alpha_M \cap (-) : H^k(M, \partial M) \longrightarrow H_{n-k}(M)$$

is an isomorphism.

First, we show that $\partial M \subseteq M$ admits a "collar neighborhood."

<u>Prop</u> Let M be a compact manifold with boundary. There is an open subset $\partial M \subseteq \mathcal{U}$ and a homeomorphism

$$(\mathcal{U}, \partial M) \cong (\partial M \times [0,1), \partial M \times \{0\}).$$



<u>Rmk</u> The statement is true more generally for paracompact M.

For the proof, we use a partition of unity.

Lemma Let M be a compact orientable manifold with boundary. The connecting homomorphism

$$\delta : H_n(M, \partial M) \to H_{n-1}(\partial M)$$

is an isomorphism.

Proof Consider the exact sequence

$$H_n(M) \to H_n(M, \partial M) \xrightarrow{\delta} H_{n-1}(\partial M) \to H_{n-1}(M).$$

Since $M \simeq \mathring{M}$, the first term vanishes, so δ is injective, and the last term is torsion-free, so δ is surjective. $\square$