Last time

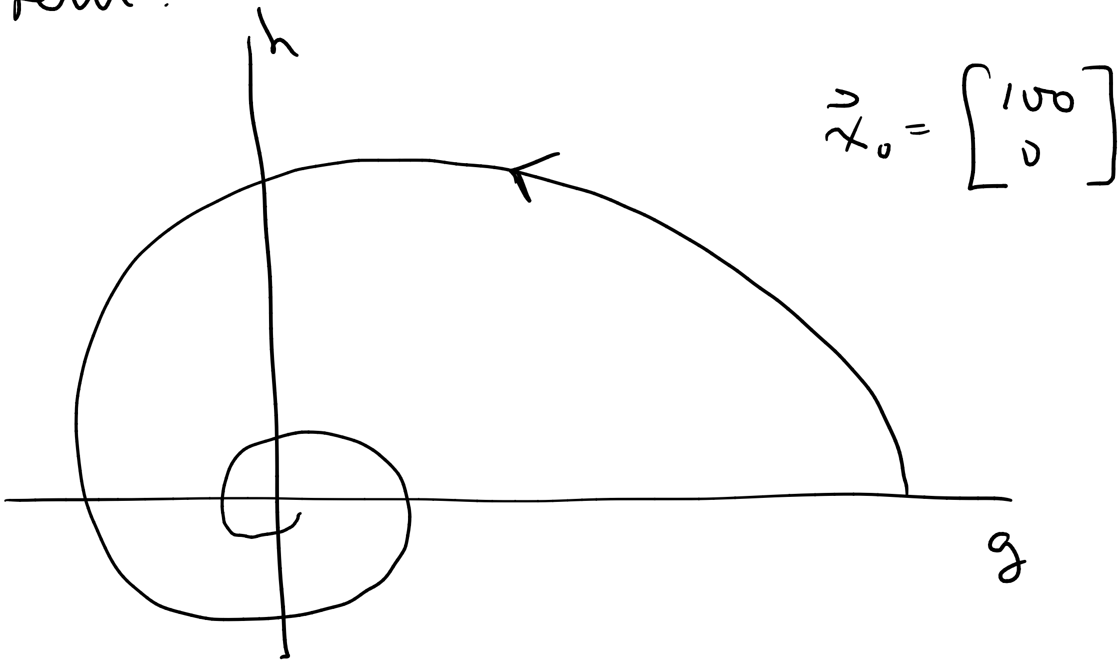
- Discrete dynamical systems $\vec{x}_{t+1} = A\vec{x}_t$
- Exponential growth/decay, equilibria
- $A = SDS^{-1} \Rightarrow A^t = SD^tS^{-1}$

Ex $A = \frac{1}{10}\begin{bmatrix} 9 & -4 \\ 1 & 9 \end{bmatrix}$

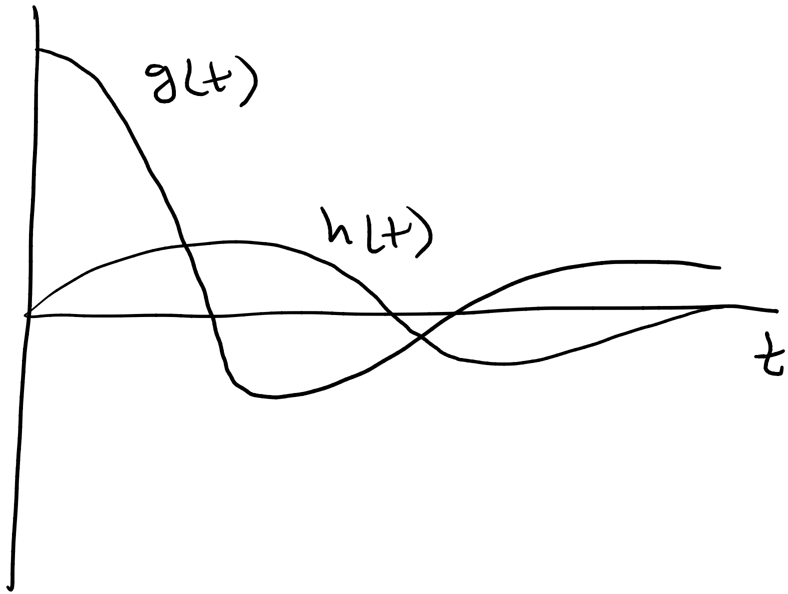
Again, the eigenvalues will be $1/10$ of the roots of

$$\det\begin{bmatrix} 9-\lambda & -4 \\ 1 & 9-\lambda \end{bmatrix} = \lambda^2 - 18\lambda + 85$$

Since $b^2-4ac = 324 - 340 = -16 < 0$, there are no roots. Plotting values reveals a new, oscillatory behavior of the system:



Or, as functions of t:



To see how this new behavior is connected to the failure of our previous approach, we adjust our perspective: $f_A(\lambda) = \lambda^2 - 1.8\lambda + 0.85$ <u>does</u> have roots, so A <u>does</u> have eigenvalues!

$$\lambda = \frac{1}{10} \frac{18 \pm \sqrt{-16}}{2}$$

$$= \frac{1}{10}(9 \pm 2i),$$

where i is a symbol such that $i^2 = -1$. These are <u>complex numbers</u>.

Complex numbers $z = a + bi$, $a + b$ real

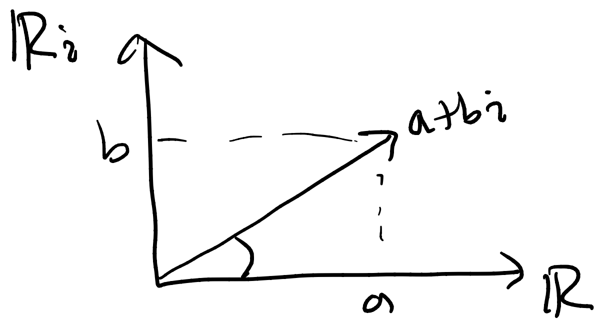
(0) $i^2 = -1$

(1) a and b are the real and imaginary parts, respectively

(2) $(a+bi) + (c+di) = a+c + (b+d)i$

(3) $(a+bi)(c+di) = ac - bd + (ad+bc)i$

(4) The complex plane $\mathbb{C}$:



(5) Polar form: $z = r(\cos\theta + i\sin\theta)$,

where $r = \sqrt{a^2+b^2}$

$$\cos\theta = \frac{a}{r}$$

$$\sin\theta = \frac{b}{r}$$

(6) If $w = s(\cos\varphi + i\sin\varphi)$, then

$$zw = rs(\cos(\theta+\varphi) + i\sin(\theta+\varphi)).$$

(7) $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

(8) If $p(\lambda)$ is a polynomial with complex coefficients, then

$$p(\lambda) = k(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$$

Linear algebra works the same for matrices with complex entries, so we can diagonalize "over $\mathbb{C}$".

<u>Ex</u> $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$f_A(\lambda) = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$, $\lambda = \pm i$

$E_i = \ker \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} = \ker \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} i \\ 1 \end{bmatrix}$

$E_{-i} = \ker \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = \ker \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Ex More generally, the rotation-and-scaling matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is diagonalizable with

$$S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}.$$

Surprisingly, this tells us something strong about a general 2×2 matrix:

If A is a real 2×2 matrix with eigenvalues $a \pm bi$ $(b \neq 0)$ and $\vec{v} + i\vec{w}$ is an eigenvector with $\lambda = a+bi$, then

$$S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

So every such matrix is similar to a rotation and scaling matrix. Informally, complex eigenvalues capture oscillatory behavior.

<u>Ex</u> Returning to our example,

$$A = \frac{1}{10}\begin{bmatrix} 9 & -4 \\ 1 & 9 \end{bmatrix}, \quad \lambda = \frac{1}{10}(9 \pm 2i)$$

$$E_+ = \ker\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} = \ker\begin{bmatrix} 1 & -2i \\ 0 & 0 \end{bmatrix} = \mathrm{Span}\begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}} + i\underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{w}}$$

$$S^{-1}AS = \frac{1}{10}\begin{bmatrix} 9 & -2 \\ 2 & 9 \end{bmatrix} \qquad S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix $\frac{1}{10}\begin{bmatrix} 9 & -2 \\ 2 & 9 \end{bmatrix}$ represents scaling by $\frac{1}{10}\sqrt{81+4} \approx 0.92$ and rotation by $\theta \approx 0.2$ radians, so $S^{-1}A^tS = (S^{-1}AS)^t$ represents scaling by $\approx 0.92^t$ and rotation by $\approx 0.2t$ radians. In particular, $\lim_{t\to\infty} A^t = 0$, and we have explained the "spiraling inward" picture from before.