Last time

- Non-embedding
- Manifolds with boundary
- Collars
- Lefschetz duality

Correction:

__Corollary__ If $X \subseteq \mathbb{R}^n$ is compact and locally contractible, then $H_i(X)$ is trivial for $i \geq n$ and free Abelian for $i = n-1$ __and__ $n-2$.

__Rmk__ We avoid the general UCT as long as we can.

Proof By compactness and local contractibility, $H_i(X)$ is finitely generated for every i, and AD gives $H^{\geq n}(X)=0$ and $H^{n-1}(X)$ free. From the LES in cohomology for (p a prime)

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{F}_p \to 0,$$

we extract the SES

$$0 \to \operatorname{coker}(H^i(X) \overset{p}{\to}) \to H^i(X;\mathbb{F}_p) \to \ker(H^{i+1}(X) \overset{p}{\to}) \to 0,$$

which gives $\dim H^i(X;\mathbb{F}_p) = \operatorname{rk} H^i(X)$ for $i \geq n-1$

By finite generation, this number is equal to $\dim H_i(X;\mathbb{F}_p)$. The LES in homology gives

$$0 \to \operatorname{coker}(H_i(X) \overset{p}{\to}) \to H_i(X;\mathbb{F}_p) \to \ker(H_{i-1}(X) \overset{p}{\to}) \to 0.$$

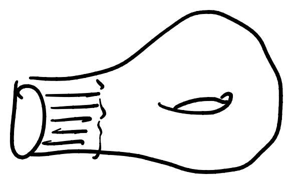
For $i \geq n$, it follows that $H_i(X)$ is finite without p^2-torsion and $H_{i-1}(X)$ has no p-torsion. Since p was arbitrary, it follows that $H_i(X) = 0$ for $i \geq n$ and $H_{i-1}(X)$ is free Abelian. To show that H_{n-2} is free Abelian, we appeal (for now) to the following general result. $\square$

Prop Suppose $H_*(X)$ is of finite type, so $H_i(X) \cong \mathbb{Z}^{\beta_i(X)} \oplus T_i$ with T_i finite. Then $H^i(X) \cong \mathbb{Z}^{\beta_i(X)} \oplus T_{i-1}$.

We return to the main thread.

Prop Let M be a compact manifold with boundary. There is an open subset $\partial M \subseteq U$ and a homeomorphism

$$(U, \partial M) \cong (\partial M \times [0,1), \partial M \times \{0\}).$$



Thm Let $\{U_i\}_{i=1}^n$ be an open cover of the normal space X. There exist $\varphi_i : X \to [0,1]$ for $1 \le i \le n$ such that

(1) $\overline{\varphi_i^{-1}((0,1])} \subseteq U_i$ for $1 \le i \le n$, and

(2) $\sum_{i=1}^n \varphi_i \equiv 1$.

Proof of prop Since M is compact Hausdorff, hence normal, so is ∂M. By compactness, $\partial M = \bigcup_{i=1}^n U_i$ with $U_i = V_i \cap \partial M$ for open $\mathbb{R}^n_+ \cong V_i \subseteq M$. Choose a partition of unity $\{\varphi_i\}_{i=1}^n$ on ∂M as above.

Setting $\psi_k = -\sum_{i=1}^{k} \varphi_i$, consider the subspaces

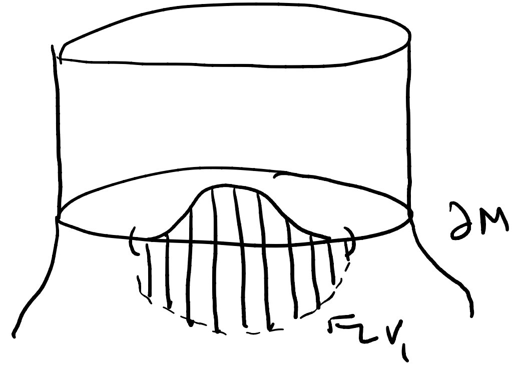
$M_k \subseteq M \coprod_{\partial M} \partial M \times [-1,0]$ given by

$$M_k = M \cup \{(x,t) \mid t \geq \psi_k(x)\}.$$

Then $M_0 = M$ and M_n is a manifold satisfying the conclusion, so it suffices to construct homeomorphisms

$$f_k : M_{k-1} \xrightarrow{\cong} M_k.$$

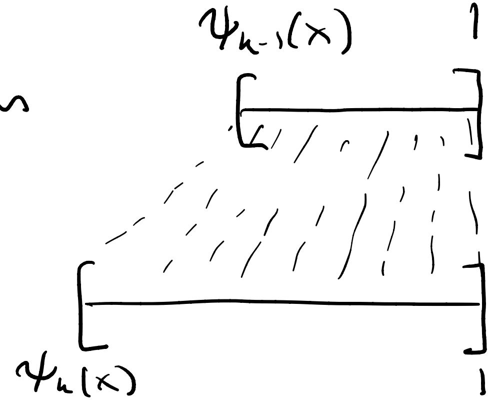
Via the homeomorphism $V_k \cong \mathbb{R}^n_+$ and inclusion



$\mathcal{U}_k \times [-1,0] \subseteq M_n$, we obtain an embedding $\mathcal{U}_k \times [-1,1] \subseteq M_n$. We define $f_k\big|_{M_{k-1} \cap \mathcal{U}_k \times [-1,1]}$ by

$$f_k(x,t) = \left(x, \frac{1-\psi_k(x)}{1-\psi_{k-1}(x)} \big(t - \psi_{k-1}(x)\big) + \psi_k(x) \right),$$

which lies in M_k, and we let f_k be the inclusion $M_{k-1} \subseteq M_k$ away from $\mathcal{U}_k \times [-1,1]$. Since



$\psi_{k-1} = \psi_k$ outside of $\overline{\varphi_k^{-1}((0,1])} \subseteq U_k$, f_k is well-defined and continuous, and its inverse may be defined similarly.

$\square$

<u>Proof of lemma</u> We have

$$
\begin{aligned}
H^k(M,\partial M) &\cong H^k(M, \partial M \times [0,1]) \\
&\cong H^k(\mathring{M}, \partial M \times (0,1)) \\
&\cong \varinjlim H^k(\mathring{M}, \partial M \times (0, 1/N)) \\
&\cong H^n_c(\mathring{M}) \\
&\cong H_{n-k}(\mathring{M})
\end{aligned}
$$

$$\cong \begin{cases} \mathbb{Z} & k=n \\ 0 & k>n \end{cases}.$$

As before, it follows that $H_n(M,\partial M) \cong \mathbb{Z}$.

$\square$