Last time

- Corollary revisited
- Collars exist
- $H_n(M, \partial M) \cong \mathbb{Z}$

To prove Lefschetz duality, we require one further ingredient.

<u>Prop</u> Let M be a manifold and $Z \subseteq M$ compact and locally contractible. There is a LES

$$\cdots \to H^k_c(M \setminus Z) \to H^k_c(M) \to H^k(Z) \to H^{k+1}_c(M \setminus Z) \to \cdots$$

__Exercise__ Given subspaces $B \subseteq A \subseteq X$, there is a long exact sequence

$$\cdots \to H^k(X,A) \to H^k(X,B) \to H^k(A,B) \to H^{k+1}(X,A) \to \cdots,$$

and similarly for homology.

__Exercise__ For directed sets I and J and directed system $\{A_{ij}\}_{(i,j)\in I\times J}$, there are natural isomorphisms

$$\varinjlim_I \varinjlim_J A_{ij} \cong \varinjlim_{I\times J} A_{ij} \cong \varinjlim_J \varinjlim_I A_{ij}$$

<u>Proof of prop</u> Given a compact subset $K \subseteq M$ and open $Z \subseteq \mathcal{U} \subseteq M$ with compact closure, consider the exact sequence

$$\cdots \to H^k(M, M \setminus K \cup \mathcal{U}) \to H^k(M, M \setminus K) \to H^k(M \setminus K \cup \mathcal{U}, M \setminus K) \to \cdots$$

We pass to the limit indexed over K and $\mathcal{U}$ and identify terms. The third term above is isomorphic to $H^k(\mathcal{U}, M \setminus K \cap \mathcal{U})$ by excision. For $\mathcal{U}$ fixed, every K is contained in the compact $K \cup \overline{\mathcal{U}}$, so the limit over K of this term is simply $H^k(\mathcal{U})$. Thus, by the exercise, the limiting third term is

$\varinjlim_{\mathcal{U}} H^k(\mathcal{U}) \cong H^k(Z)$, since Z is locally contractible and M embeds in Euclidean space (deformed). The first term above is isomorphic to $H^k(M \setminus Z, (M \setminus K \cup \mathcal{U}) \setminus Z)$. We may take the limit over pairs $(K, \mathcal{U})$ with $\mathcal{U} \subseteq K$. We observe:

(1) $(M \setminus K \cup \mathcal{U}) \setminus Z$ is the complement in $M \setminus Z$ of $K \setminus \mathcal{U}$, which is compact.

(2) Every compact subset of $M \setminus Z$ is of this form.

Thus, the limiting term here is $H^k_c(M \setminus Z)$.

$\square$

<u>Proof of LD</u> Many elements of this argument are familiar, so we will be brief.

Consider the diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(M) \arrow[r, "i^*"] \arrow[d, dashed, "\cong"'] & H^k(\partial M) \arrow[r, "\delta"] \arrow[d, "D", "\cong"'] & H^{k+1}(M,\partial M) \arrow[r] \arrow[d, "\alpha_M \cap (-)"] & H^{k+1}(M) \arrow[r] \arrow[d, dashed, "\cong"'] & \cdots \\
\cdots \arrow[r] & H_{n-k}(M,\partial M) \arrow[r, "\delta"] & H_{n-k-1}(\partial M) \arrow[r, "i_*"] & H_{n-k-1}(M) \arrow[r] & H_{n-k-1}(M,\partial M) &
\end{tikzcd}$$

By the five lemma and induction, it suffices to exhibit the dashed isomorphisms and check commutativity up to sign. For the first task, we apply the proposition with

$$Z = M \setminus \partial M \times [0, 1/2) \subseteq \mathring{M}$$

$$\simeq M$$

$$\mathring{M} \setminus Z = \partial M \times (0, 1/2)$$

to obtain the commuting diagrams with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(\partial M \times (0,1/2)) \arrow[r] \arrow[d, "D", "\cong"'] & H^k_c(\mathring{M}) \arrow[r] \arrow[d, "D", "\cong"'] & H^k(M) \arrow[r] \arrow[d, dashed] & \cdots \\
\cdots \arrow[r] & H_{n-k}(\partial M) \arrow[r] & H_{n-k}(\mathring{M}) \arrow[r] & H_{n-k}(M, \partial M) \arrow[r] & \cdots
\end{tikzcd}$$

By the five lemma, the dashed arrow is an isomorphism. For commutativity, we say only that all vertical arrows are induced by capping with a "fundamental class" in each setting, and the isomorphism

$$H_n(M, \partial M) \xrightarrow[\cong]{\delta} H_{n-1}(\partial M)$$

shows that a fundamental class for M is sent to one for ∂M. □

For the same reason as before, LD holds over $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ with the same hypotheses and with $\operatorname{char}\mathbb{F} = 2$ without assuming orientability. We take coefficients in $\mathbb{F}$ in the following.

<u>Cor</u> Let $\mathbb{F}$ be a field and N a compact $(2k+1)$-manifold with $\partial N = M$, oriented unless $\operatorname{char}\mathbb{F} = 2$. Then $\dim H_k(M)$ is even and

$$\frac{1}{2}\dim H^k(M) = \operatorname{rk}\left(H^k(N) \xrightarrow{i^*} H^k(M)\right).$$

Proof Consider the diagram

$$\begin{tikzcd}
H^k(N) \arrow[r, "i^*"] & H^k(M) \arrow[r, "\delta"] \arrow[d, "\cong"'] & H^{k+1}(N,M) \arrow[d, "\cong"] \\
 & H_k(M) \arrow[r, "i_*"] & H_k(N)
\end{tikzcd}$$

By exactness,

$$\begin{aligned}
\operatorname{rk} i^* &= \dim \ker \delta \\
&= \dim \ker i_* \\
&= \dim H_k(M) - \operatorname{rk} i_* \\
&= \dim H_k(M) - \operatorname{rk} i^*. \qquad \square
\end{aligned}$$

<u>Cor</u> If M is a connected, compact manifold, then $\chi(M)$ is

(1) zero if $\dim M$ is odd

(2) even if M bounds.

<u>Cor</u> $\mathbb{R}P^{2k}$ and $\mathbb{C}P^{2k}$ do not bound.

<u>Proof</u> $\chi(\mathbb{R}P^{2k}) = 1$ and $\chi(\mathbb{C}P^{2k}) = 2k+1$. $\square$