Last time

- LES in H_c^*
- Proof of Lefschetz duality
- Middle dimensional homology
- $\mathbb{RP}^{2k}$ and $\mathbb{CP}^{2k}$ do not bound

Cor If M is a connected, compact manifold, then $\chi(M)$ is

(1) zero if $\dim M$ is odd

(2) even if M bounds.

<u>Exercise</u> For any X with $H_*(X)$ finitely generated. For any prime p,

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim H_i(X; \mathbb{F}_p)$$

<u>Proof of corollary</u> If $n = \dim M$ is odd, then

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H_i(M; \mathbb{F}_2)$$

$$= \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \dim H_i(M; \mathbb{F}_2)$$

$$+ \sum_{i=\frac{n+1}{2}}^{n} (-1)^i \dim H^{n-i}(M; \mathbb{F}_2)$$

$$= \left(\text{—} \;\; '' \;\; \text{—} \right)$$

$$+ \sum_{i=0}^{\frac{n-1}{2}} (-1)^{n-i} \dim H_i(M; \mathbb{F}_2) \Big)$$

$$= 0 .$$

For the second claim, we may take $\dim M = 2k$ by (1), in which case the same reasoning shows that

$$\chi(M) = (-1)^k \dim H_k(M),$$

which is even by the previous result.

$\square$

As these results make clear, the middle degree (co)homology of a manifold is special.

<u>Recall</u> For M a compact, connected, orientable $2k$-manifold, the bilinear form

$$H^k(M;\mathbb{R}) \otimes H^k(M;\mathbb{R}) \xrightarrow{-\cup-} H^{2k}(M;\mathbb{R}) \xrightarrow{D} \mathbb{R}$$

is non-degenerate. If k is even, this pairing is symmetric, hence represented by a unique invertible symmetric matrix, diagonalizable by the spectral theorem.

Def The signature of a symmetric bilinear form is the sum of the signs of the eigenvalues of the associated symmetric matrix. The signature $\sigma(M)$ of M is 0 if $4 \nmid \dim M$ and otherwise it is the signature of the duality pairing above.

Thm If M bounds a compact orientable manifold, then $\sigma(M) = 0$.

<u>Proof</u> WLOG $\dim M = 2k$ with k even. Writing $i: M = \partial N \hookrightarrow N$, recall that $\dim H^k(M;\mathbb{R}) = 2\,\mathrm{rk}\, i^* =: 2m$. We claim that the composite

$$\operatorname{im} i^* \otimes \operatorname{im} i^* \to H^k(M;\mathbb{R}) \otimes H^k(M;\mathbb{R}) \xrightarrow{\smile} H^{2k}(M;\mathbb{R})$$

vanishes. Assuming so, suppose that the symmetric matrix in question has r positive eigenvalues, hence $2m-r$ negative eigenvalues. Then there is a subspace $W^+ \subseteq H^k(M;\mathbb{R})$ with $\dim W^+ = r$ such

$Q|_{W^+}$ is positive definite, where Q is the associated quadratic form (resp. W^-, $2m-r$, negative definite). From the definition of Q and the claimed vanishing, $\operatorname{im} i^* \cap W^+ = 0$, so

$$m + r = \operatorname{rk} i^* + \dim W^+ \leq \dim H^k(M;\mathbb{R}) = 2m,$$

whence $r \leq m$. For the same reason, $2m - r \leq m$, so $r \geq m$. Thus, $r = m = 2m - r$, so $\sigma(M) = r - (2m-r) = 0$.

To prove the claim, we return to the diagram

$$\begin{array}{ccccc} H^k(N) & \xrightarrow{i^*} & H^k(M) & \xrightarrow{\delta} & H^{k+1}(N, M) \\ & & \downarrow \cong & & \downarrow \cong \\ & & H_k(M) & \xrightarrow{i_*} & H_k(N) \end{array}$$

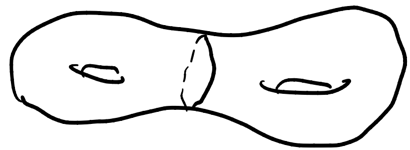
Given $\alpha, \beta \in H^k(N)$, we have $\delta(i^*\alpha \cup i^*\beta) = 0$ by exactness, since i^* is a ring homomorphism. But δ is injective in degree n since i_* is injective in degree 0. $\square$

<u>Def</u> Let M_j be connected, oriented n-manifolds. The <u>connected sum</u> $M_1 \# M_2$ is defined as follows: choosing orientation preserving embeddings $i_j : \mathbb{R}^n \to M_j$, define

$$M_1 \# M_2 = \frac{M_1 \setminus i_1(\mathring{D}^n) \amalg M_2 \setminus i_2(\mathring{D}^n)}{\sim},$$

where $i_1(S^n)$ is identified with $i_2(S^n)$ via an orientation reversing homeomorphism.

Although it is not immediately obvious, this operation is well-defined up to homeomorphism

<u>Ex</u> $T^2 \# T^2 \cong \Sigma_2$ 

<u>Exercise</u> The connected sum is again orientable, and the quotient map $M_1 \# M_2 \to M_1 \vee M_2$ induces isomorphisms on H_i for $0 < i < n$.

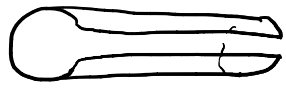
MV

<u>Cor</u> With the previous assumptions, there is an isomorphism of rings

$$H^*(M_1 \# M_2) \cong \frac{H^*(M_1) \times H^*(M_2)}{\langle (1,-1), (\alpha_{M_1}, -\alpha_{M_2}) \rangle}$$

Ex $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not bound an orientable 5-manifold. Indeed, the cohomology ring is generated by $x, y \in H^2$ with the relation $x^2 = y^2$, so the symmetric matrix in question is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\sigma(\mathbb{CP}^2 \# \mathbb{CP}^2) = 2 \neq 0$.

Ex Writing $-\mathbb{CP}^2$ for the opposite orientation, $\mathbb{CP}^2 \# -\mathbb{CP}^2$ bounds the orientable 5-manifold $\mathbb{CP}^2 \setminus \mathring{D}^2 \times [0,1]$



This is not a contradiction, since the cohomology ring here instead has the relation $x^2 = -y^2$, so the signature vanishes.