Big idea To understand an object, probe it with simpler objects.

Q (Topology 2) What do simplices know about a space?

A Homology and cohomology.

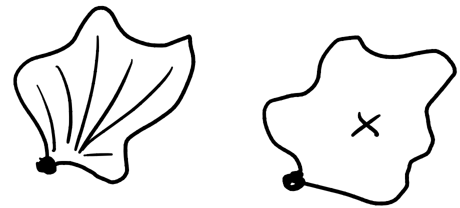
Q What do spheres know about a space?

Notation Given pointed spaces (X, x_0) and (Y, y_0) write,

$$[X, Y] = \{ f : (X, x_0) \to (Y, y_0) \} / \sim,$$

where $f \sim g$ iff f and g are homotopic through pointed maps.

Notation We set $\pi_n(X, x_0) := [S^n, X]$, where the basepoint of S^n is the image of $\partial D^n \subseteq D^n \to S^n$.



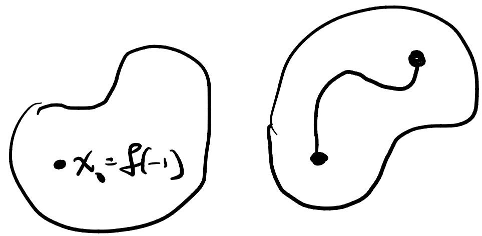
A0 Path components

Ex For any $x_0 \in X$, the function

$$[S^0, X] \longrightarrow \left\{ \begin{array}{c} \text{path components} \\ \text{of } X \end{array} \right\} =: \pi_0(X)$$

$$[f] \longmapsto f(1)$$

is a bijection.

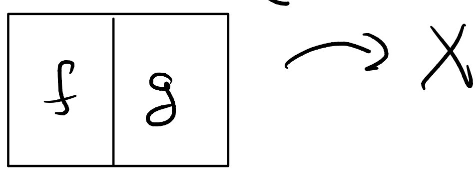


A1–∞ Something new???

In our study of homology, we were forced to introduce addition by force. Here, this is unnecessary. We restrict to $n > 0$.

<u>Construction</u> Identify S^n with $[0,1]^n/\partial$. Given $[f], [g] \in \pi_n(X, x_0)$, define $[f] \star [g]$ to be the equivalence class of

$$(f \star g)(t_1, \dots, t_n) := \begin{cases} f(2t_1, t_2, \dots, t_n) & t_1 \in [0, 1/2] \\ g(2t_1 - 1, t_2, \dots, t_n) & t_1 \in [1/2, 1] \end{cases}$$



<u>Exercise</u> The operation $\star$ is well-defined on π_n.

Convexity lemma Given convex $K_2 \subseteq \mathbb{R}^n$ and $A \subseteq K_1$, any two maps $f, f' : K_1 \to K_2$ with $f|_A = f'|_A$ are homotopic rel A, i.e., via a homotopy H with $H(a,-)$ constant for $a \in A$

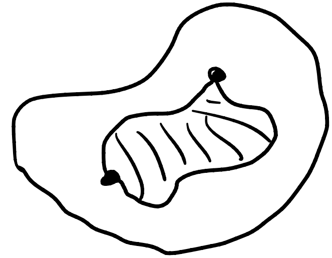
Proof The straight-line homotopy

$$H(x,t) = tf(x) + (1-t)f'(x)$$

is well-defined by convexity and has the desired property. $\square$

Definition Paths $\gamma_1, \gamma_2 : [0,1] \to X$ are <u>path homotopic</u> if they are homotopic via a homotopy constant on $\{0,1\}$ (a "path homotopy"). We write $f \simeq_p g$.

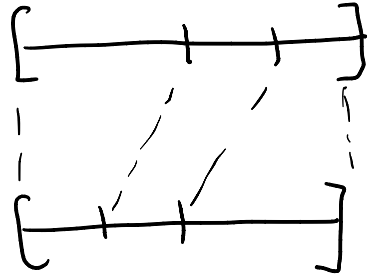
<u>Rmk</u> $\pi_1(X, x_0)$ is the set of path homotopy classes of loops.



<u>Cor</u> Paths in a convex space are path homotopic iff they have the same endpoints.

<u>Prop / Def</u> For $n > 0$, $(\pi_n(X, x_0), *)$ is a group, called the n^{th} <u>homotopy group</u> of X (at x_0).

Proof Let $L : [0,1] \to [0,1]$ be the unique piecewise linear map mapping $[0,1/2]$, $[1/2,3/4]$ and $[3/4,1]$ onto $[0,1/4]$, $[1/4,1/2]$, and $[1/2,1]$, respectively. Then the following diagram commutes



$$\begin{tikzcd}
{[0,1]\times[0,1]^{n-1}} \arrow[d, "L\times \mathrm{id}"'] \arrow[dr, bend left, "f_1\star(f_2\star f_3)"] & \\
{[0,1]\times[0,1]^{n-1}} \arrow[r, bend right, "(f_1\star f_2)\star f_3"'] & X
\end{tikzcd}$$

and $L \simeq_p \mathrm{id}_{[0,1]}$ by convexity, implying associativity. The same argument (exercise) shows that the class of the constant map is a unit for $\star$ and the class of the reverse

$$\bar{f}(t_1, \dots, t_n) = f(1-t_1, \dots, t_n)$$

is an inverse for $[f]$. $\square$

Note that, for $n > 1$, we could have defined $\star$ using a different coordinate.

Prop (Eckmann-Hilton argument) Let X be a set with two unital binary operations $\star$ and $\circ$. If $(x \circ y) \star (z \circ w) = (x \star z) \circ (y \star w)$ for $x, y, z, w \in X$, then $\circ$ and $\star$ are equal, commutative, and associative.

Proof First, the units $1_\star$ and $1_\circ$ coincide, as

$$\begin{aligned} 1_\circ &= 1_\circ \circ 1_\circ \\ &= (1_\star \star 1_\circ) \circ (1_\circ \star 1_\star) \\ &= (1_\star \circ 1_\circ) \star (1_\circ \circ 1_\star) \end{aligned}$$

$$= 1_\star \star 1_\star$$

$$= 1_\star .$$

Next, given $x, y \in X$,

$$x \circ y = (1_\star \star x) \circ (y \star 1_\star)$$

$$= (1_\star \circ y) \star (x \circ 1_\star)$$

$$= (1_\circ \circ y) \star (x \circ 1_\circ)$$

$$= y \star x$$

$$= (y \circ 1_\circ) \star (1_\circ \circ x)$$

$$= (y \star 1_\circ) \circ (1_\circ \star x)$$

$$= (y \star 1_\star) \circ (1_\star \star x)$$

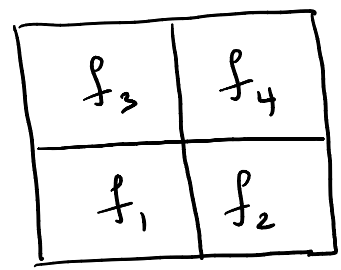
$$= y \circ x,$$

establishing equality and commutativity. Associativity, which we do not use, is an exercise.

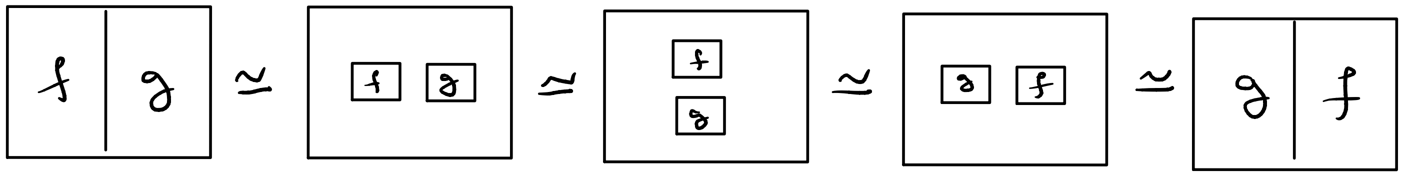
$\square$

<u>Cor</u> For $n>1$, the group $\pi_n(X, x_0)$ is Abelian.

<u>Proof</u> It is an easy exercise to check the hypothesis of the EH argument. $\square$



Alternative picture proof of commutativity:



Another difference from homology is the involvement of the basepoint.

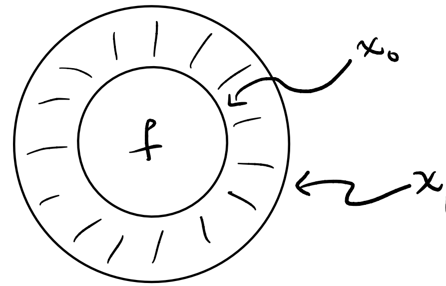
<u>Prop</u> Given a path α from x_0 to x_1 in X, there is a canonical isomorphism

$$\hat{\alpha} : \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_1).$$

<u>Proof</u> Viewing S^n as $D^n/\partial D^n$, and given

$[f] \in \pi_n(X, x_0)$, define $\hat{\alpha}([f])$ to be the class of

$$\hat{\alpha}(f)(v) = \begin{cases} f(2v) & |v| \le 1/2 \\ \alpha(2|v|-1) & |v| \ge 1 \end{cases}$$



It is an exercise to check that $\hat{\alpha}$ is a well-defined homomorphism depending only on the path homotopy class of α. The latter statement implies that $\hat{\alpha}$ is an isomorphism with inverse $\hat{\bar{\alpha}}$ (using convexity). $\square$

Exercise Restricting to α with $\alpha(0)=\alpha(1)$, this construction defines an action of $\pi_1(X,x_0)$ on $\pi_n(X,x_0)$, which for $n=1$ is conjugation.

When X is path connected, therefore, we write $\pi_n(X) := \pi_n(X,x_0)$.

As with homology, a continuous map $f: X \to Y$ induces a homomorphism

$$f_*: \pi_n(X,x_0) \longrightarrow \pi_n(Y,f(x_0))$$
$$[g] \longmapsto [f\circ g],$$

and this structure is again "functorial" in that it respects composition and identity.

<u>Non-proposition</u> Homotopic maps induce the same homomorphism on homotopy groups.

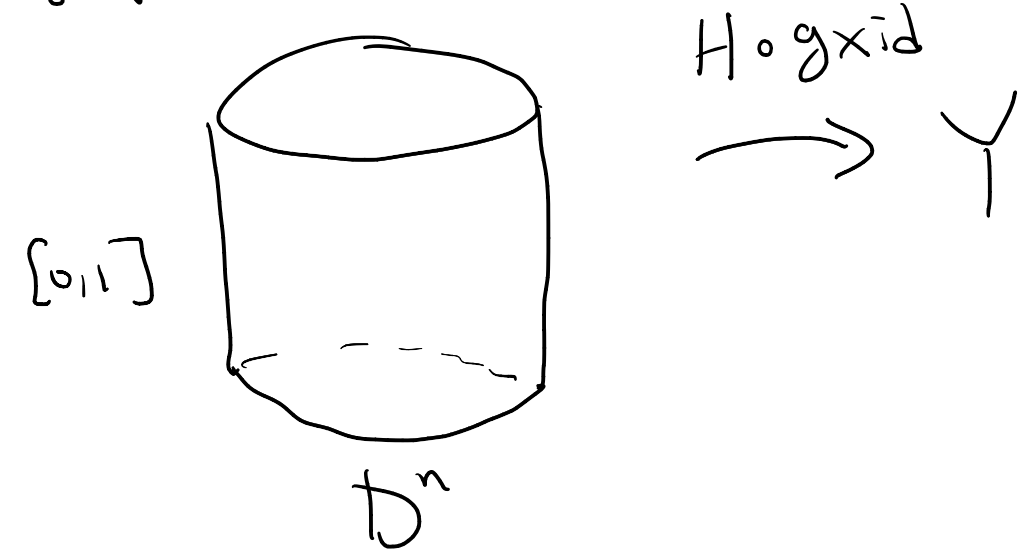
What's wrong? In most cases, f_* + g_* don't even have the same target!

<u>Prop</u> Let $f, f' : X \to Y$ be maps and choose a basepoint $x_0 \in X$. Given a homotopy H from f to f', define $\alpha : [0,1] \to Y$ by

$$\alpha(t) = H(x_0, t).$$

Then $f'_* = \hat{\alpha} \circ f_*$.

<u>Proof</u> Choose $[g] \in \pi_n(X, x_0)$ and consider the composite map



On the bottom face of the cylinder, this map is $f \circ g$ (resp. top, $f' \circ g$), and it is α on every vertical segment. The result follows by applying convexity to two appropriate maps $(D^n, \partial D^n) \longrightarrow (D^n \times [0,1], \partial D^n \times \{1\})$.

$\square$

Cor If $f: X \to Y$ is a homotopy equivalence, then $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism.

Proof Given a homotopy inverse g, since $g \circ f \simeq id$, we have

$$g_* \circ f_* = (g \circ f)_* = \hat{\alpha} \circ (id)_* = \hat{\alpha} \circ id,$$

so $\hat{\alpha}^{-1} \circ g_*$ is a left inverse for f_*, and similarly for $f \circ g$. $\square$