Last time

- Homotopy groups
- Convexity + Eckmann-Hilton
- Group structure, functoriality, homotopy invariance

Skepticism 0 What's up with π_1 vs. π_n?

Skepticism 1 How different are homotopy groups from homology groups?

Skepticism 2 Even if they are different, can we ever compute them?

Skepticism 3 Even if we can compute them, is it worth it?

The first part of the course will be aimed at addressing these skepticisms.

To begin we set ourselves the following goal.

Goal) Calculate $\pi_i(S^n)$ for $n \geq 0$

Def We say X is <u>simply connected</u> if

$|\pi_0(X)| = |\pi_1(X)| = 1$.

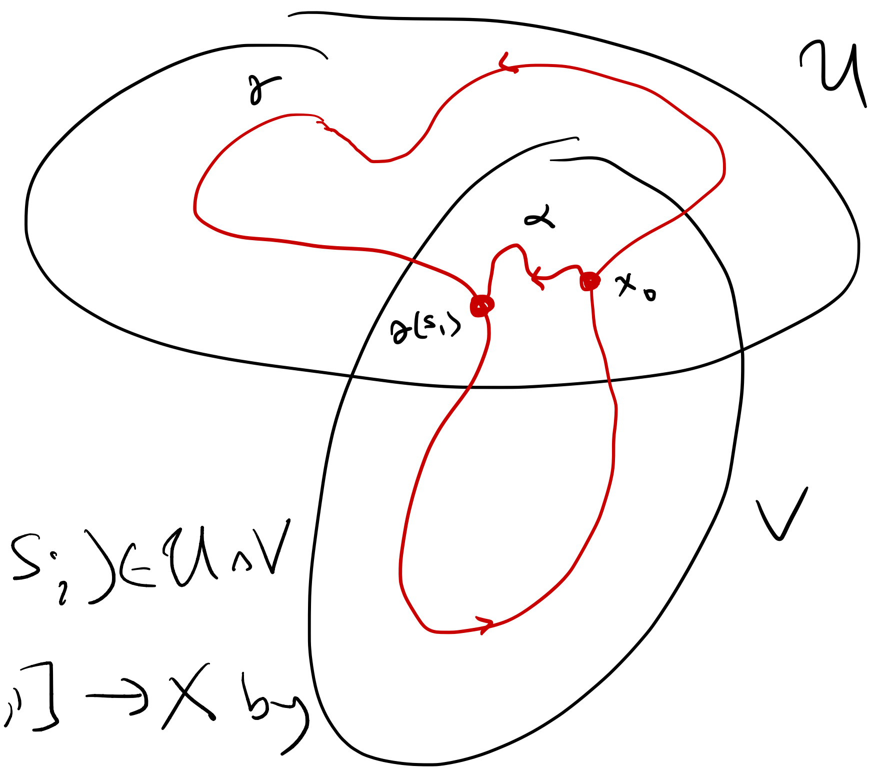
<u>Theorem</u> Let $i: U \hookrightarrow X$ and $j: V \hookrightarrow X$ be inclusions of open subspaces with $U \cap V$ path connected. If $X = U \cup V$, then $\pi_1(X, x_0)$ is generated by $\mathrm{im}(i_*)$ and $\mathrm{im}(j_*)$ for any $x_0 \in U \cap V$.

Corollary If $X = U \cup V$ with U and V open and simply connected and $U \cap V$ non-empty and path connected, then X is simply connected.

Corollary For $n > 1$, S^n is simply connected.

Proof of thm Fix $[\gamma] \in \pi_1(X, x_0)$.

By the Lebesgue number lemma there exist $0 = s_0 < s_1 < \dots < s_r = 1$ such that $\gamma([s_{i-1}, s_i])$ lives in U or in V and $\gamma(s_i) \in U \cap V$ for $1 \le i \le r$. Define $\sigma_i : [0,1] \to X$ by



$$\gamma_i(s) = \gamma\Big((s_i - s_{i-1})s + s_{i-1}\Big),$$

and choose paths α_i from x_0 to $\gamma(s_i)$ in $U \cap V$ for each $0 \le i \le r$ with $\alpha_0 = e_{x_0} = \alpha_r$ (we use that $U \cap V$ is path connected). By convexity of $[0,1]$,

$$[\gamma] = [\gamma_1 * \cdots * \gamma_r]$$
$$= [\alpha_0 * \gamma_1 * \overline{\alpha_1} * \alpha_1 * \cdots * \overline{\alpha_{r-1}} * \alpha_{r-1} * \gamma_r * \overline{\alpha_r}]$$
$$= \prod_{i=1}^{r} [\alpha_{i-1} * \gamma_i * \overline{\alpha_i}].$$

Since $[\alpha_{i-1} * \gamma_i * \overline{\alpha_i}]$ lies either in $\operatorname{im}(i_*)$ or in $\operatorname{im}(j_*)$, it follows that $[\gamma]$ lies in the subgroup they generate. Since γ was arbitrary, the proof is complete. $\square$

<u>Observation</u> Loops in S^1 lift to paths in $\mathbb{R}$.

Indeed, given a covering map $p: E \to B$, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$, unique path lifting determines the function

$$\gamma \longmapsto \tilde{\gamma}(1)$$

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{loops in } B \\ \text{at } b_0\end{array}\right\} \arrow[r] \arrow[d] & p^{-1}(b_0) \\
\pi_1(B, b_0) \arrow[ur, dashed, "\exists ?"'] &
\end{tikzcd}$$

Thm (Lifting correspondence) Let $p: E \to B$ be a covering map, fix b_0 and $e_0 \in p^{-1}(b_0)$, and set $H = p_*(\pi_1(E, e_0))$. The lifting correspondence

$$\pi_1(B, b_0)/H \longrightarrow p^{-1}(b_0)$$
$$H[\gamma] \longmapsto \tilde{\gamma}(1)$$

is well-defined and injective. It is surjective if E is path connected.

Remark The lifting correspondence depends on the choice of $e_0 \in p^{-1}(b_0)$, and it is not a homomorphism ($p^{-1}(b_0)$ has no group structure).

Cor If E is path connected, then
$$|p^{-1}(b_0)| = [\pi_1(B,b_0):H].$$

Cor If E is simply connected, then
$$|p^{-1}(b_0)| = |\pi_1(B,b_0)|.$$

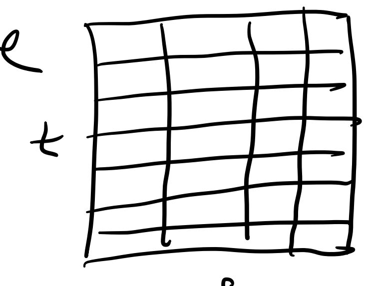
Cor For $n>1$, $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$

Cor $\pi_1(S^1) \cong \mathbb{Z}$

Proof The lifting correspondence for $1 \in S^1$ and $0 \in \mathbb{R}$ gives a bijection, through which $\pi_1(S^1)$ inherits a second group structure, to which Eckmann-Hilton applies (exercise). $\square$

Prop Let $p: E \to B$ be a covering map and H a path homotopy in B with $H(0,0) = b_0$. For every $e_0 \in p^{-1}(b_0)$, there is a unique lift $\tilde{H}$ of H with $\tilde{H}(0,0) = e_0$, and $\tilde{H}$ is also a path homotopy.

The proof is very similar to the argument for path lifting, applying the Lebesgue number lemma to find a fine subdivision of $[0,1] \times [0,1]$ instead, and defining the lift piece by piece. The details are an exercise.



Proof of theorem Suppose H is a path homotopy from γ to γ'. Then $\tilde{H}$ is also a path homotopy, and $\tilde{H}(-,0)$ is a lift of γ starting at e_0, hence

equal to $\tilde{\gamma}$ by uniqueness (resp. $\tilde{H}(s,1)$, $\tilde{\gamma}'$).
Thus, $\tilde{\gamma} \simeq_p \tilde{\gamma}'$, so $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$. Thus, the correspondence descends to $\pi_1(B, b_0)$.

Next, given loops γ_1 and γ_2 at b_0, we claim that $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$ iff $[\gamma_1][\gamma_2]^{-1} \in H$.
The "if" direction gives well-definition on cosets, and the "only if" direction gives injectivity.

If $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$, then $\tilde{\gamma}_1 \star \overline{\tilde{\gamma}_2}$ is defined and is a loop at e_0 lifting $\gamma_1 \star \overline{\gamma_2}$, so

$$H \ni p_\star([\tilde{\gamma}_1 \star \overline{\tilde{\gamma}_2}]) = [\gamma_1 \star \overline{\gamma_2}] = [\gamma_1][\gamma_2]^{-1}!$$

Conversely, suppose that $[\gamma_1][\gamma_2]^{-1} \in H$. Then $[\gamma_1] = [\delta \star \gamma_2]$ for some loop δ with $\tilde{\delta}(1) = e_0$.

Thus, $\tilde{\gamma}_1(1) = \widetilde{\delta \star \gamma_2}(1) = \tilde{\delta} \star \tilde{\gamma}_2(1) = \tilde{\gamma}_2(1)$.

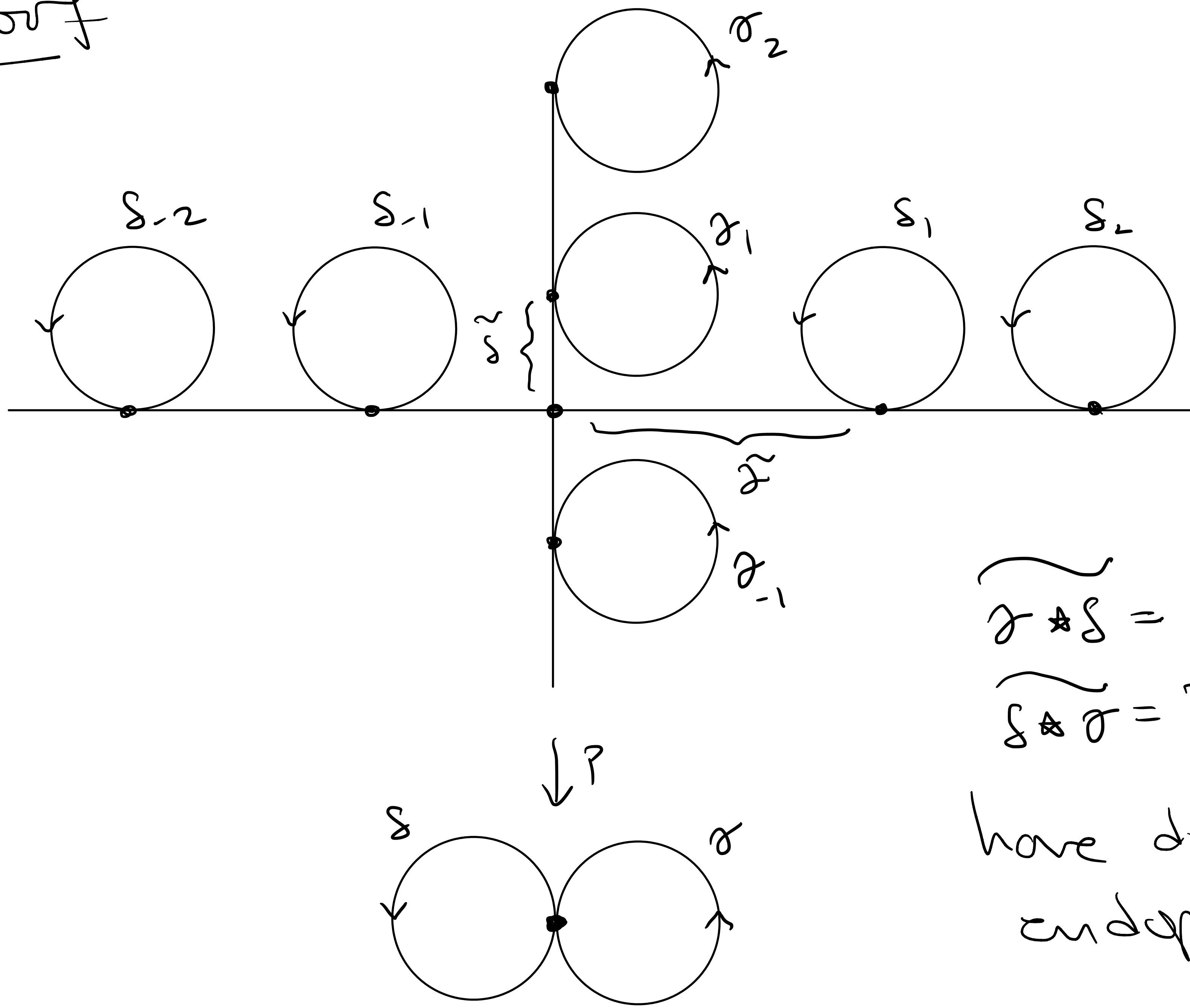
Finally, if E is path connected, there is a path α from e_0 to any other $e_1 \in p^{-1}(b_0)$. Then $p \circ \alpha$ is a loop in B such that

$$[p \circ \alpha] \longmapsto e_1$$

under the lifting correspondence. $\square$

Cor $\pi_1(S^1 \vee S^1)$ is non-Abelian.

Proof



$$\widetilde{\gamma \star \delta} = \tilde{\gamma} \star \delta_1$$

$$\widetilde{\delta \star \gamma} = \tilde{\delta} \star \gamma_1$$

have different endpoints.

$\square$