Last time

- Gluing simple connectivity
- Lifting correspondence
- $\pi_i(S^n)$, $n>0$
- $\pi_1(S^1 \vee S^1)$ non-Abelian

In particular $\pi_1 \not\cong H_1$. Nevertheless, the two are connected.

Thm (Hurewicz) For X path connected and $x_0 \in X$, there is a canonical isomorphism

$$\pi_1(X, x_0)^{ab} \xrightarrow{\cong} H_1(X),$$

which is natural for maps respecting basepoints.

To produce the homomorphism, we begin by observing that the source and target are built from the same data. There is a standard linear homeomorphism $[0,1] \cong \Delta^1$ (which we suppress), leading to the leftmost bijection in the diagram

$$
\begin{tikzcd}
\left\{\begin{matrix}\text{paths}\\ \text{in } X\end{matrix}\right\} \arrow[rr, two heads] \arrow[d, equal, "\wr"'] & & \left\{\begin{matrix}\text{paths}\\ \text{in } X\end{matrix}\right\}\Big/\sim_p \arrow[r, phantom, "\supseteq"] \arrow[d, dashed, "\eta''"'] & \pi_1(X,x_0) \arrow[r, two heads] \arrow[d, dashed, "\eta'"'] & \pi_1(X,x_0)^{ab} \arrow[dl, dashed, "\eta"'] \\
\left\{\begin{matrix}\text{singular}\\ \text{1-simplices}\\ \text{of } X\end{matrix}\right\} \arrow[r, phantom, "\subseteq"] & C_1(X) \arrow[r, two heads] & C_1(X)\big/B_1 \arrow[r, phantom, "\supseteq"] & H_1(X) &
\end{tikzcd}
$$

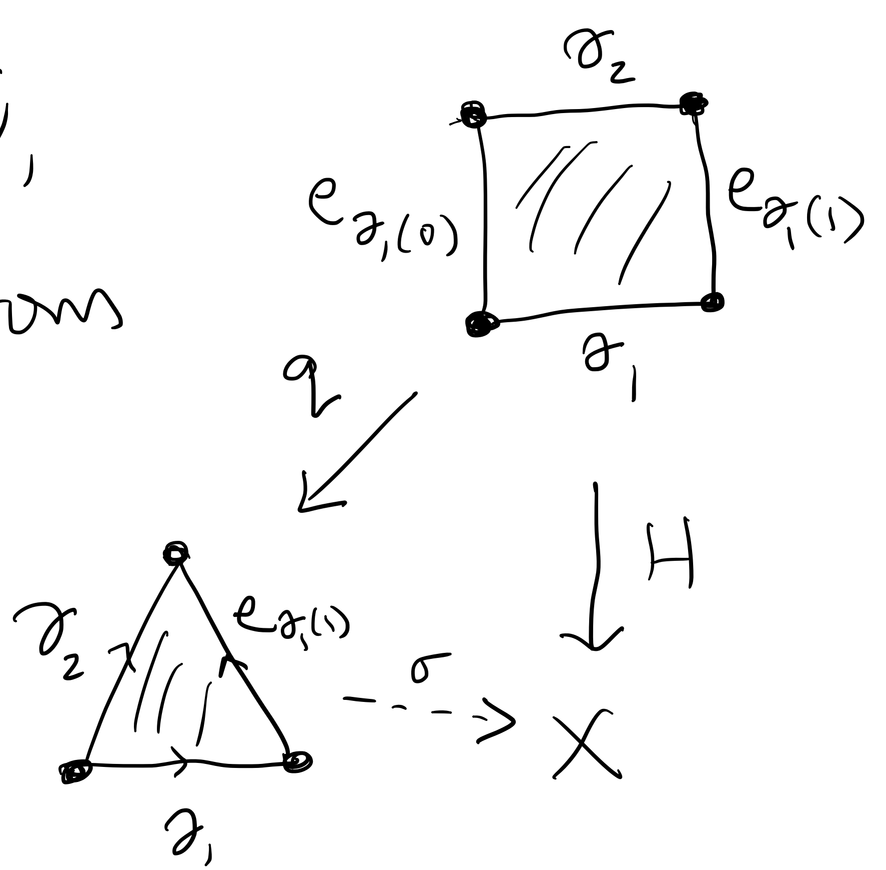
Lemma 0 The constant path e_x at $x \in X$ is a boundary for any $x \in X$.

Proof It is the boundary of the constant 2-simplex at x. $\square$

Lemma 1 $\gamma_1 \sim_p \gamma_2 \Rightarrow \gamma_1 - \gamma_2 \in B_1$.

Proof Given a path homotopy $H : [0,1] \times [0,1] \to X$, since $H\big|_{\{0\} \times [0,1]}$ is constant, H factors through the quotient $q : [0,1] \times [0,1] \to \Delta^2$ collapsing this edge to a point. This factorization σ has the property that

$$\partial\sigma = e_{\gamma_1(1)} - \gamma_2 + \gamma_1,$$



so the claim follows from Lemma 0. $\square$

Thus, η'' exists. Since a loop is obviously a cycle, it follows that η' exists — at the level of sets!

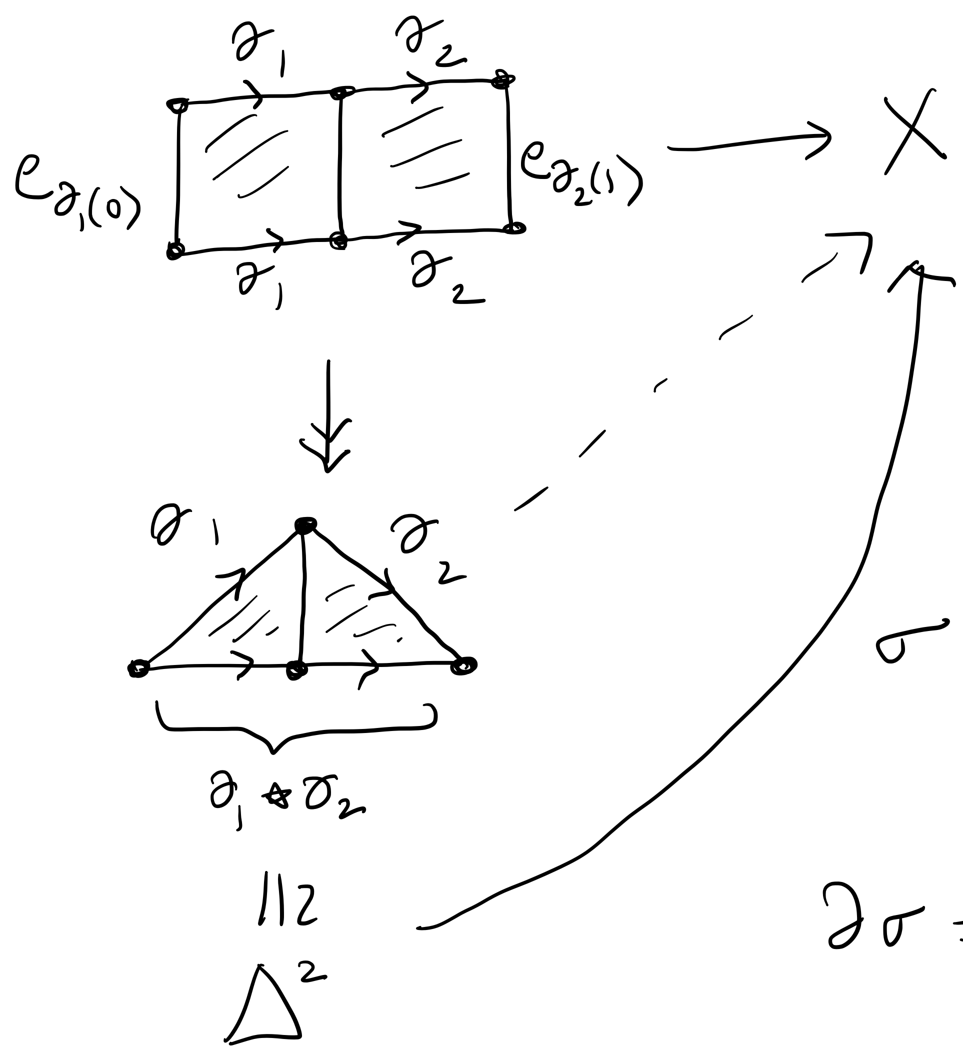
<u>Lemma 2</u> η' is a homomorphism.

The existence of η follows from Lemma 2 and the universal property of Abelianization.

<u>Lemma 3</u> For any γ_1 and γ_2 with $\gamma_2(0) = \gamma_1(1)$,

$$\gamma_1 \star \gamma_2 - \gamma_1 - \gamma_2 \in B_1$$

<u>Proof</u>



$$\partial \sigma = \gamma_2 - \gamma_1 \star \gamma_2 + \gamma_1$$

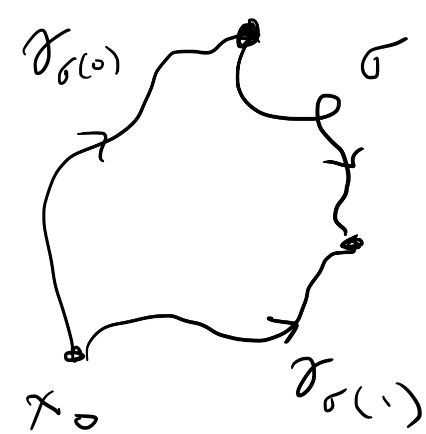
$\square$

Lemma 2 follows immediately from Lemma 3, so we obtain the homomorphism

$$\eta : \pi_1(X, x_0)^{ab} \longrightarrow H_1(X)$$
$$[\gamma] \longmapsto \gamma \bmod B_1$$

To conclude, we will construct the inverse of η. For every $x \in X$, choose a path γ_x from x_0 to x arbitrarily, except that $\gamma_{x_0} = e_{x_0}$ (here we use the assumption on X). The assignment

$$\sigma \longmapsto [\gamma_{\sigma(0)} * \sigma * \overline{\gamma_{\sigma(1)}}]$$



gives the first function in the diagram

$$
\begin{tikzcd}
\left\{\begin{array}{c}\text{singular}\\ \text{1-simplices}\\ \text{of } X\end{array}\right\} \arrow[r, phantom, "\subseteq"] \arrow[d, "\nu''"] & C_1(X) \arrow[r, phantom, "\supseteq"] \arrow[drr, dashed, "\nu'"] & Z_1 \arrow[r, two heads] & H_1(X) \arrow[d, dashed, "\nu"] \\
\left\{\begin{array}{c}\text{loops in } X\\ \text{at } x_0\end{array}\right\} \arrow[rr, two heads] & & \pi_1(X, x_0) \arrow[r, two heads] & \pi_1(X, x_0)^{ab}
\end{tikzcd}
$$

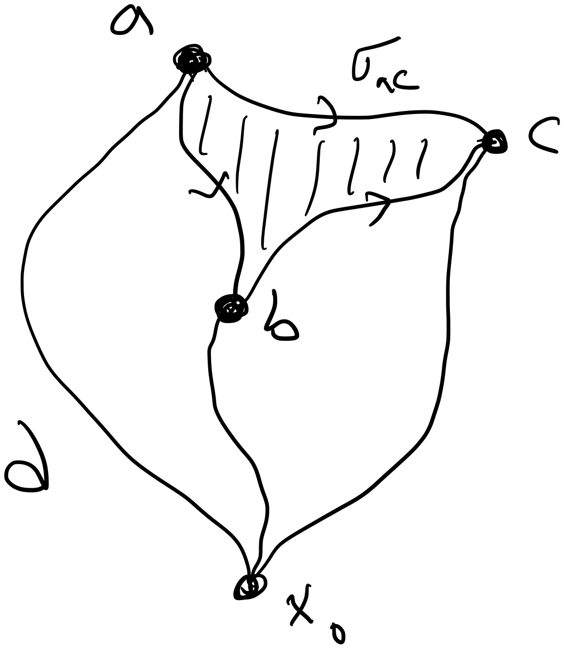
The existence of ν' follows from the universal property of the free Abelian group and the fact that $\pi_1(X, x_0)^{ab}$ is Abelian.

For the existence of ν, we have the following.

<u>Lemma 4</u> $\nu'(B_1) = 0$

<u>Proof</u> Given $\sigma : \Delta^2 \to X$, write

$a = \sigma(1,0,0)$, $b = \sigma(0,1,0)$, $c = \sigma(0,0,1)$.



Then σ restricts to paths between these points, denoted σ_{ab}, etc. Then

$$\nu'(\partial\sigma) = \nu'(\sigma_{bc} - \sigma_{ac} + \sigma_{ab})$$

$$= [\gamma_b * \sigma_{bc} * \overline{\gamma_c}] - [\gamma_a * \sigma_{ac} * \overline{\gamma_c}] + [\gamma_a * \sigma_{ab} * \overline{\gamma_b}]$$

$$= [\gamma_b * \sigma_{bc} * \overline{\gamma_c} * \gamma_c * \overline{\sigma_{ac}} * \overline{\gamma_a} * \gamma_a * \sigma_{ab} * \overline{\gamma_b}]$$

$$= [\gamma_b * \sigma_{bc} * \overline{\sigma_{ac}} * \sigma_{ab} * \overline{\gamma_b}]$$

$$= [\gamma_b * \overline{\gamma_b}]$$

$$= 0,$$

where we have used that $\sigma_{bc} * \overline{\sigma_{ac}} * \sigma_{ab}$ is path nullhomotopic, since Δ^2 is convex.

$\square$

<u>Proof of theorem</u> Given a loop γ, we have

$$\begin{aligned} \nu(\eta([\gamma])) &= [\nu''(\gamma)] \\ &= [\gamma_{x_0} * \gamma * \overline{\gamma_{x_0}}] \\ &= [e_{x_0} * \gamma * \overline{e_{x_0}}] \\ &= [\gamma]. \end{aligned}$$

On the other hand, given a cycle $c = \sum_{i=1}^{m} n_i \sigma_i$

$$\eta(\nu(c)) = \eta\left(\sum_{i=1}^{m} n_i\left[\gamma_{\sigma_i(0)} \star \sigma_i \star \overline{\gamma_{\sigma_i(1)}}\right]\right)$$

$$= \sum_{i=1}^{m} n_i\, \gamma_{\sigma_i(0)} \star \sigma_i \star \overline{\gamma_{\sigma_i(1)}} \mod B_1$$

(Lemma 3) $$= \sum_{i=1}^{m} n_i\left(\gamma_{\sigma_i(0)} + \sigma_i + \overline{\gamma_{\sigma_i(1)}}\right) \mod B_1$$

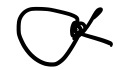
(Lemma 3, exercise) $$= \sum_{i=1}^{m} n_i\left(\gamma_{\sigma_i(0)} + \sigma_i - \gamma_{\sigma_i(1)}\right) \mod B_1$$

but $\sum_{i=1}^{m} n_i\left(\gamma_{\sigma_i(0)} - \gamma_{\sigma_i(1)}\right) = 0$, since $\partial c = 0$.

Thus, ν is both left and right inverse to η.

$\square$

Q How to calculate $\pi_1(\infty)$?

Idea $\infty = U_1 \cup U_2$, where $U_1 =$  and $U_2 =$ .

Since U_1 and U_2 deformation retract onto a circle and $U_1 \cap U_2$ onto a point, we should be able to calculate $\pi_1(\infty)$ in terms of $\mathbb{Z}$ and the trivial group.

Clue If each U_i is open and $U_1 \cap U_2$ is path connected, then the dashed filler in

$$\begin{tikzcd}
 & \pi_1(U_1, x_0) \arrow[dr, "(i_1)_*"] \arrow[drr, bend left, "\varphi_1"] & & \\
\pi_1(U_1 \cap U_2, x_0) \arrow[ur, "(j_1)_*"] \arrow[dr, "(j_2)_*"'] & & \pi_1(X, x_0) \arrow[r, dashed, "\varphi"] & G \\
 & \pi_1(U_2, x_0) \arrow[ur, "(i_2)_*"'] \arrow[urr, bend right, "\varphi_2"'] & &
\end{tikzcd}$$

is unique if it exists by generation.