Last time

- Hurewicz: $\pi_1(X,x_0)^{ab} \cong H_1(X)$
- Toward $\pi_1(S^1 \vee S^1)$

Thm (Seifert-van Kampen) Suppose that $X = U_1 \cup U_2$ with $U_1, U_2 \subseteq X$ open and U_1, U_2 and $U_1 \cap U_2$ all path connected. Given a group G and the solid commuting diagram of homomorphisms

$$\begin{tikzcd}
& \pi_1(U_1, x_0) \arrow[dr, "(i_1)_*"] \arrow[drr, bend left, "\varphi_1"] & & \\
\pi_1(U_1 \cap U_2, x_0) \arrow[ur, "(j_1)_*"] \arrow[dr, "(j_2)_*"'] & & \pi_1(X, x_0) \arrow[r, dashed, "\varphi"] & G \\
& \pi_1(U_2, x_0) \arrow[ur, "(i_2)_*"'] \arrow[urr, bend right, "\varphi_2"'] & &
\end{tikzcd}$$

(i.e., such that $\varphi_1 \circ (j_1)_* = \varphi_2 \circ (j_2)_*$), the dashed filler φ (i.e., such that $\varphi \circ (i_1)_* = \varphi_1$ and $\varphi \circ (i_2)_* = \varphi_2$) exists and is unique, where $x_0 \in U_1 \cap U_2$

We have already proven uniqueness, and the same result tells us how to define φ.

<u>Def</u> A (path/loop) <u>factorization</u> of $[\gamma]$ of <u>length</u> r is an r-tuple $((\gamma_1, a_1), \ldots, (\gamma_r, a_r))$, where

(1) $a_k \in \{1,2\}$

(2) γ_k is a (path/loop at x_0) lying entirely in U_{a_k}

(3) $\gamma \simeq_p \gamma_1 \star \cdots \star \gamma_k$.

Given a loop factorization of $[\gamma]$, $\varphi([\gamma])$ is determined by the requirements that $\varphi_0(i_1)_\star = \varphi_1$, $\varphi_0(i_2)_\star = \varphi_2$ and φ is a homomorphism:

$$\varphi([\gamma]) = \varphi_{a_1}([\gamma_1]) \cdots \varphi_{a_k}([\gamma_k])$$

Since every $[\gamma]$ admits a loop factorization by our earlier generation result, it suffices to verify that this expression is independent of the choice of factorization.

<u>Def</u> Path factorizations are <u>path equivalent</u> if they differ by a finite sequence of the following operations:

(A) exchange $((\gamma_1, a_1), \ldots, (\gamma_r, a_r))$ with $((\gamma_1, a_1), \ldots, (\gamma_{k-1} * \gamma_k, a_k), \ldots, (\gamma_r, a_r))$ if $a_{k-1} = a_k$,

(B) exchange $(\gamma_k, 1)$ with $(j_2 \circ \gamma_k', 2)$ if $\gamma_k = j_1 \circ \gamma_k'$ for some $\gamma' : [0,1] \to U_1 \cap U_2$,

(C) replace (γ_k, a_k) with (γ_k'', a_k) if $\gamma_k \simeq_p \gamma_k''$ via a path homotopy lying in U_{a_k}.

Two loop factorizations are <u>loop equivalent</u> if they are path equivalent through loop factorizations.

<u>Lemma 1</u> Any two path factorizations of $[\gamma]$ are path equivalent.

<u>Lemma 2</u> If loop factorizations are path equivalent, then they are loop equivalent.

<u>Proof of thm</u> By the lemmas, it suffices to show that the definition of φ is invariant under the three moves between loop factorizations.

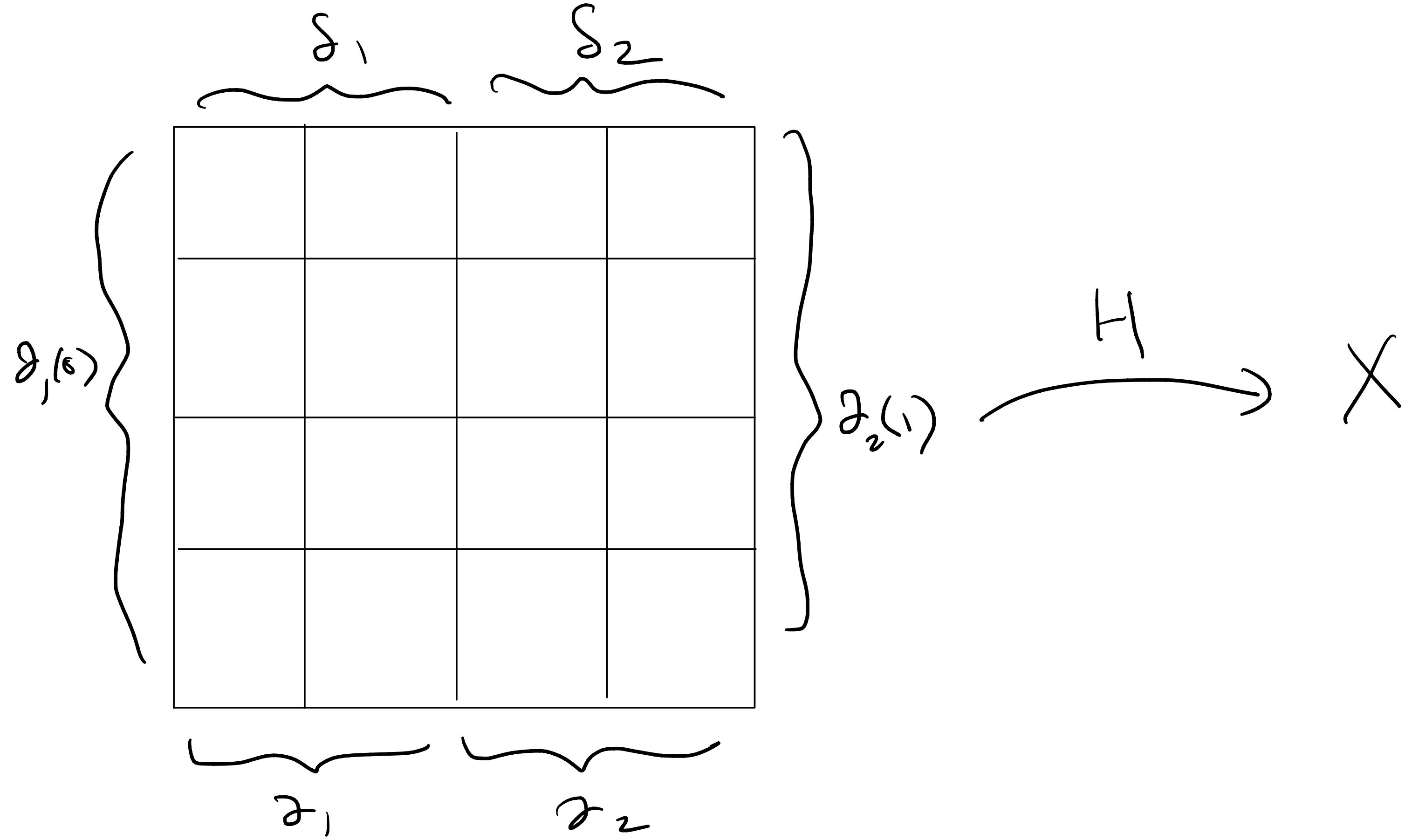
For (A), we use that φ_{a_k} is a homomorphism, for (B) we use that $\varphi_{a_1} \circ (j_1)_* = \varphi_{a_2} \circ (j_2)_*$, and for (C) the claim is immediate. $\square$

<u>Lemma 3</u> A factorization of length r is equivalent to some factorization of length R for any $R \geq r$.

<u>Proof</u> The factorization $\Big((\gamma_1, a_1), \ldots, (\gamma_r, a_r), \underbrace{(e_{x_0}, a_r), \ldots, (e_{x_0}, a_r)}_{R-r}\Big)$ is equivalent to $\big((\gamma_1, a_1), \ldots, (\gamma_r, a_r)\big)$ via $R-r$ operations of type (A) and one of type (C). $\square$

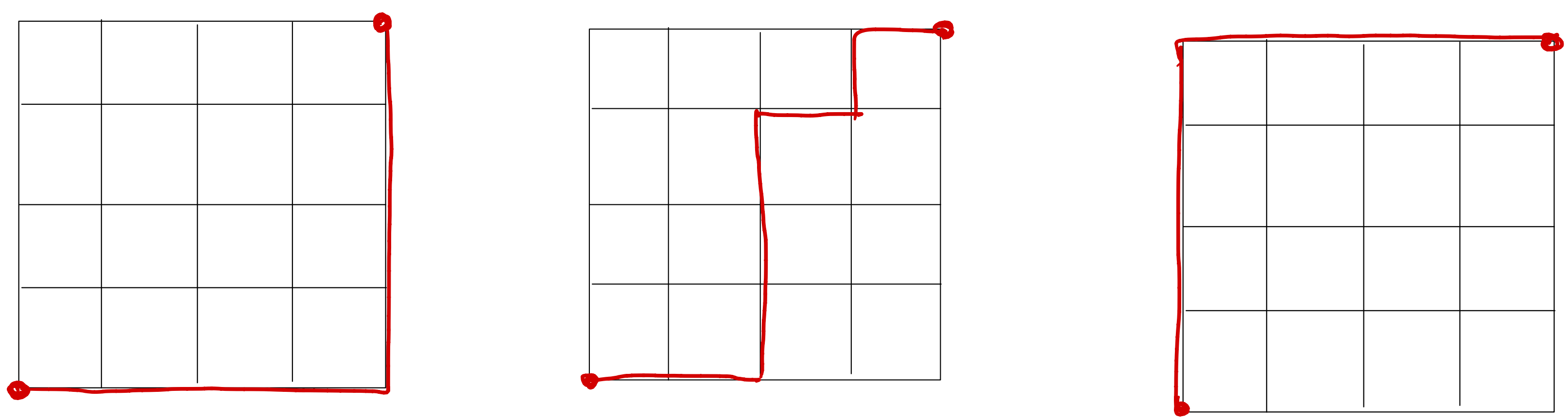
<u>Proof of Lemma 1</u> WLOG the two factorizations have the same length, say $\gamma_1 * \cdots * \gamma_r$ and $\delta_1 * \cdots * \delta_r$. Choose a path homotopy H

between them, and subdivide $[0,1]\times[0,1]$ into squares of side $1/N$ with image under H entirely in $\mathcal{U}_1$ or $\mathcal{U}_2$ (apply the LNL to $\{H^{-1}(\mathcal{U}_1), H^{-1}(\mathcal{U}_2)\}$. Without loss of generality, $r \mid N$.

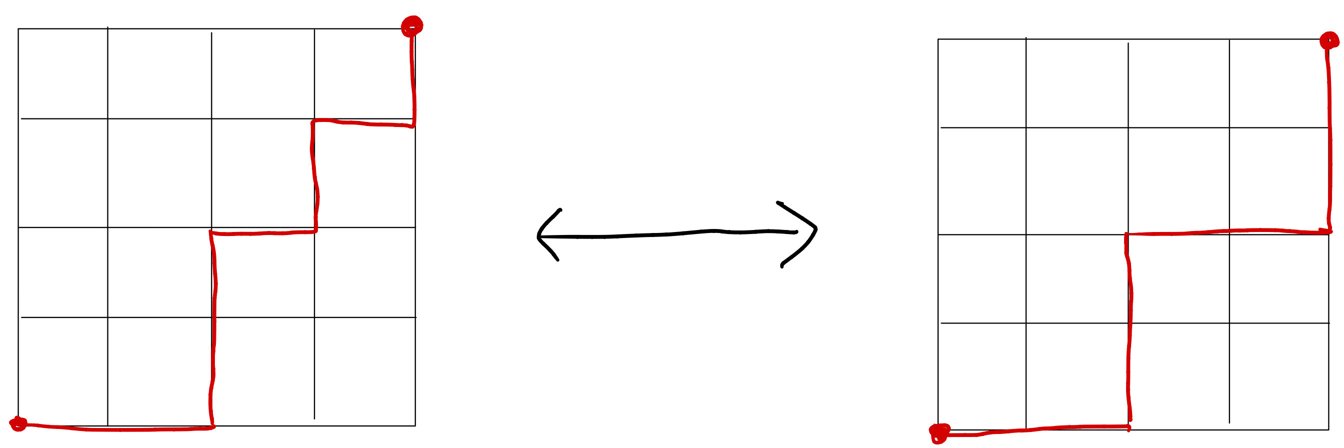


$\begin{pmatrix} N=4 \\ r=2 \end{pmatrix}$

Every non-decreasing path of edges from $(0,0)$ to $(1,1)$ determines a path factorization of the same homotopy class (we use convexity), well-defined up to (B) moves:



The factorizations corresponding to the two paths around the boundary differ from our original factorizations by (A) and (C) moves. The claim follows upon observing that the factorizations corresponding to any two paths of edges differ by (C) and (potentially) (B) moves (again by convexity):



$\square$

The proof of Lemma 2 is left as an exercise.

The theorem characterizes $\pi_1(X, x_0)$ by a universal property.

Exercise A group satisfying the conclusion of SVK is unique up to unique isomorphism under $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$.

Rmk This universal property is that of the "pushout in the category of groups".

We now make the theorem more explicit.

<u>Prop</u> Given the solid commuting diagram of homomorphisms

$$\begin{tikzcd}
 & G_1 \arrow[dr] \arrow[drr, bend left, "\varphi_1"] & & \\
H \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & & G_1 \underset{H}{*} G_2 \arrow[r, dashed, "\varphi"] & G \\
 & G_2 \arrow[ur] \arrow[urr, bend right, "\varphi_2"'] & &
\end{tikzcd}$$

the dashed filler exists and is unique.

<u>Proof</u> Ignoring H, the universal property of the free group produces the unique homomorphism

$$\begin{tikzcd}
G_1 \arrow[dr, "i_1"] \arrow[drr, bend left, "\varphi_1"] & & \\
 & G_1 * G_2 \arrow[r, dashed, "\tilde{\varphi}"] & G \\
G_2 \arrow[ur, "i_2"] \arrow[urr, bend right, "\varphi_2"'] & &
\end{tikzcd}$$

But $\ker(\tilde{\varphi}) \supseteq N$, where N is the normal closure of $\{\psi_1(h)\psi_2(h)^{-1} \mid h \in H\}$.

$$\tilde{\varphi}(\psi_1(h)\psi_2(h)^{-1}) = \tilde{\varphi}(\psi_1(h))\,\tilde{\varphi}(\psi_2(h))^{-1}$$
$$= \varphi_1(\psi_1(h))\,\varphi_2(\psi_2(h))^{-1}$$
$$= 1$$

by commutativity. The universal property of the quotient provides the unique homomorphism

$$\begin{tikzcd} G_1 * G_2 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & G \\ G_1 * G_2 / N \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

. $\square$

Cor $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$, a free group on two generators.

We now have a fairly satisfactory understanding of the fundamental group. Does anything we've done so far apply to higher homotopy groups?

Observation The lifting correspondence for a path connected covering space $p: E \to B$ with fiber F gives the "exact sequence"

$$\pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B)$$

To study such sequences more generally, we imitate our two-step approach for homology.

(1) Define relative groups that obviously form a LES

(2) Interpret relative groups in good situations.

Extending our notation, given subspaces $X \supseteq A \supseteq x_0$ and $Y \supseteq B \supseteq y_0$, we write

$$[(X,A),(Y,B)] = \{f:(X,A,x_0) \to (Y,B,y_0)\}/\sim.$$

where $f \sim g$ iff $\exists H: X \times [0,1] \to Y$ such that $H(-,0) = f$, $H(-,1) = g$, $H(A \times [0,1]) \subseteq B$, and $H(x_0, -) = y_0$. We write

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}), (X, A)].$$

<u>Exercise</u> $\pi_n(X, A, x_0)$ is a group for $n > 1$ and Abelian for $n > 2$.

We refer to the n^{th} homotopy group of X <u>relative</u> to A (based at x_0).

<u>Exercise</u> An element $[f] \in \pi_n(X, A, x_0)$ is trivial iff the dashed map exists in the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "f|_{S^{n-1}}"] \arrow[d, hook] & A \arrow[d] \\ D^n \arrow[r, "f"] \arrow[ur, dashed] & X \end{tikzcd}$$

"compression criterion"

making the bottom triangle commute up to homotopy rel S^{n-1}, i.e., fixing S^{n-1} pointwise.

Thm The assignments

$$\pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0)$$

$$[f] \longmapsto [f|_{S^{n-1}}]$$

$$\pi_n(X,x_0) \longrightarrow \pi_n(X,A,x_0)$$

$$[f] \longmapsto [D^n \xrightarrow{q} S^n \xrightarrow{f} X]$$

are well-defined and homomorphisms for $n > 1$. Moreover, the sequence

$$\cdots \to \pi_n(A,x_0) \to \pi_n(X,x_0) \to \pi_n(X,A,x_0) \to \pi_{n-1}(A,x_0) \to \cdots \to \pi_0(A,x_0) \to \pi_0(X,x_0)$$

is exact.