Last time

- Van Kampen
- $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$
- Relative homotopy $\pi_n(X, A, x_0) = [(D^n, S^{n-1}), (X, A)]$.

Clarification We define $\pi_0(X, A, x_0)$ to be the quotient of $\pi_0(X, x_0)$ given by identifying the components of X that intersect A.

Clarification A sequence $(S, s_0) \xrightarrow{f} (S', s_0') \xrightarrow{g} (S'', s_0'')$ of pointed sets is exact if $g^{-1}(s_0'') = \operatorname{im}(f)$.

Thm The assignments

$$\pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0)$$

$$[f] \longmapsto [f|_{S^{n-1}}]$$

$$\pi_n(X,x_0) \longrightarrow \pi_n(X,A,x_0)$$

$$[f] \longmapsto [D^n \xrightarrow{q} S^n \xrightarrow{f} X]$$

are well-defined and homomorphisms for $n>1$. Moreover, the sequence

$$\cdots \to \pi_n(A,x_0) \to \pi_n(X,x_0) \to \pi_n(X,A,x_0) \to \pi_{n-1}(A,x_0) \to \cdots \to \pi_0(X,x_0) \to \pi_0(X,A,x_0) \to 1$$

is exact.

Proof Exactness of the sequence

$$\pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A)$$

$$\begin{array}{c} A \\ \downarrow i \\ X \end{array}$$

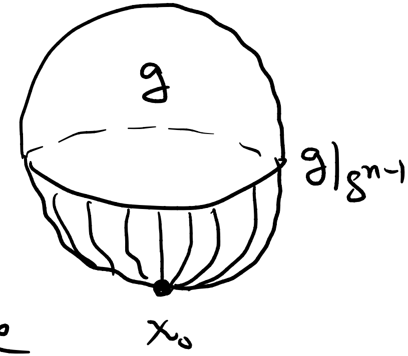
is equivalent to the compression criterion. For the sequence

$$\begin{array}{c} (X,x_0) \\ \downarrow j \\ (X,A) \end{array}$$

$$\pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A),$$

$\partial j_*[f: S^n \to X]$ is represented by the map $S^{n-1} \subseteq D^n \xrightarrow{q} S^n \xrightarrow{f} X$, which is constant, so $\operatorname{im} j_* \subseteq \ker \partial$. Conversely, supposing that $\partial[g:(D^n,S^{n-1}) \to (X,A)] = 0$, there exists a nullhomotopy of $g|_{S^{n-1}}$ with image lying in A and fixing x_0, giving the dashed filler in the commuting diagram below. The Slinky lemma then produces a map $\tilde{g}: S^n \to X$ with $j_*[\tilde{g}] = [g]$.

$$\begin{tikzcd}
D^n \arrow[dr, dashed] & \\
S^{n-1} \arrow[u, hook] \arrow[r] \arrow[d, hook] & A \arrow[d] \\
D^n \arrow[r, "g"] & X
\end{tikzcd}$$



Finally, for exactness of the sequence

$$\pi_n(X,A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X),$$

we note that $i_*\partial[f]$ is represented by either composite in the diagram below, which is nullhomotopic, since it factors through D^n, so

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & A \arrow[d] \\
D^n \arrow[r] & X
\end{tikzcd}$$

$\operatorname{im}\partial \subseteq \ker i_*$. Reversing this logic, a null-homotopy of $i \circ g$ gives an extension over D^n, defining a map on D^n restricting to g on S^{n-1}. $\square$

<u>Q</u> What does $\pi_n(X,A)$ mean?

<u>Ex</u> $H_n(X,A) \cong \tilde{H}_n(X/A)$ for a good pair (X,A).

The analogue of a good pair is given by the following extremely important definition.

Def A map $p: E \to B$ has the *homotopy lifting property* (HLP) with respect to a space X if, for every solid commuting diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d] \\ X \times [0,1] \arrow[r] \arrow[ur, dashed, "\exists"] & B, \end{tikzcd}$$

the dashed filler exists. If p has the HLP w.r.t. D^n for every $n \geq 0$, then p is a (Serre) *fibration*.

Ex A covering map has the HLP with respect to a point (actually, it is a fibration...).

Thm Let $p: E \to B$ be a fibration. For any $b_0 \in B$, $e_0 \in F := p^{-1}(b_0)$, and $n > 0$, the map

$$p_* : \pi_n(E, F, e_0) \to \pi_n(B, b_0) \qquad \begin{array}{c}(E,F)\\ \downarrow p\\ (B,b_0)\end{array}$$

is an isomorphism.

Rmk A good pair (X,A) has the homotopy extension property, which is "dual" to the HLP and makes $A \subseteq X$ a cofibration. The duality between homology/cofibrations and homotopy/fibrations is called Eckmann-Hilton duality and is a recurring theme in algebraic topology.

__Lemma__ A map $p: E \to B$ is a fibration iff it has the following extension property for all CW pairs (X, A) with A a deformation retract:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}.$$

__Proof__ Homework. $\square$

__Proof of thm__ For surjectivity, given $f: S^n \to B$, we obtain the lift

$$\begin{tikzcd} D^0 \arrow[r, "e_0"] \arrow[d] & & E \arrow[d, "p"] \\ D^n \arrow[r, "q"] \arrow[urr, dashed, "\exists \tilde{f}"] & S^n \arrow[r, "f"] & B . \end{tikzcd}$$

Since $\tilde{f}$ is a lift and $f \circ q(S^{n-1}) = \{b_0\}$, $\tilde{f}(S^{n-1}) \subseteq F$, so $\tilde{f}$ defines a map of pairs $(D^n, S^{n-1}) \xrightarrow{\tilde{f}} (E, F)$, and $p_*[\tilde{f}] = [f]$.

For injectivity, suppose $p_*[g_0] = p_*[g_1]$, so that we have the homotopy H in the diagram

$$\begin{tikzcd}
D^n \times \{0,1\} \cup D^0 \times [0,1] \arrow[rr, "\bar{g}"] \arrow[d, hook] & & E \arrow[d, "p"] \\
D^n \times [0,1] \arrow[r, two heads, "q \times \mathrm{id}"] \arrow[urr, dashed, "\exists \tilde{H}"] & S^n \times [0,1] \arrow[r, "H"] & B
\end{tikzcd}$$

where $\bar{g}|_{D^n \times \{i\}} = g_i$ and $\bar{g}(D^0 \times [0,1]) = \{e_0\}$.

The lift $\tilde{H}$ is a homotopy witnessing the equality $[g_0] = [g_1]$. $\square$

In order to use this result, we must be able to recognize fibrations.

Def A **fiber bundle** with **fiber** F is a map $p: E \to B$ such that, for every $b \in B$, there is an open neighbourhood $U \ni b$ and a homeomorphism fitting into the commuting diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\text{project}"] \\ & U & \end{tikzcd}$$

"locally trivial"

The space B is the **base space** and the space E is the **total space**.

Prop Fiber bundles are fibrations.

Ex The *trivial bundle* $B \times F \to B$ is a fiber bundle.

Cor $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$

Proof The indicated section and retraction exist in the exact sequence

$$\pi_n(X) \rightleftarrows \pi_n(X \times Y) \rightleftarrows \pi_n(Y).$$

$\square$

Ex Covering spaces are precisely the fiber bundles with discrete fibers.

Cor If $p: E \to B$ is a covering map, then $p_*: \pi_n(E) \to \pi_n(B)$ is an isomorphism for $n > 1$.

Proof $\cancelto{0}{\pi_n(F)} \to \pi_n(E) \to \pi_n(B) \to \cancelto{0}{\pi_{n-1}(F)}$.

$\square$

Cor If X has a contractible covering space, then $\pi_n(X) = 0$ for $n > 1$.

Ex $\pi_n(S^1)$ for $n > 1$.