Last time

- LES in homotopy for (X,A)
- Fibrations $\pi_n(E,F) \cong \pi_n(B)$
- Fiber bundles
- $\pi_n(S^1) = 0$ for $n>1$

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

Ex The quotient projection

$$\mathbb{C}^{n+1} \supseteq S^{2n+1} \xrightarrow{p} \mathbb{C}P^n$$

$$(z_0, \dots, z_n) \longmapsto [z_0, \dots, z_n]$$

is a fiber bundle with fiber $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$; defining $U_i = \{[z_0, \dots, z_n] \in \mathbb{C}P^n \mid z_i \neq 0\}$ for $0 \leq i \leq n$, we have the homeomorphism

$$p^{-1}(U_i) \xrightarrow{\cong} U_i \times S^1$$
$$(z_0, \dots, z_n) \longmapsto ([z_0, \dots, z_n], z_i/|z_i|).$$

In particular, we have the <u>Hopf bundle</u> $S^1 \to S^3 \to S^2$ and an S^1-bundle over $\mathbb{CP}^\infty$ with (contractible) total space S^∞.

<u>Cor</u> For $n > 0$, $\pi_n(\mathbb{CP}^\infty) \cong \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}$

<u>Proof</u> $\underbrace{\cancel{\pi_n(S^\infty)}}_{0} \to \pi_n(\mathbb{CP}^\infty) \xrightarrow{\cong} \pi_{n-1}(S^1) \to \underbrace{\cancel{\pi_{n-1}(S^\infty)}}_{0}$. $\square$

<u>Rmk</u> Contrast this calculation with the fact that $H_n(\mathbb{CP}^\infty) \neq 0$ for infinitely

many n. In particular, homology groups and homotopy groups are different.

<u>Observation</u> $\pi_n(S^n) \neq 0$ for $n > 0$. Indeed, id_{S^n} is not nullhomotopic, since S^n is not contractible.

<u>Rmk</u> Soon we will show that $\pi_n(S^n) \cong \mathbb{Z}$.

<u>Cor</u> $\pi_3(S^2) \neq 0$

<u>Proof</u>
$$\cancelto{0}{\pi_3(S^1)} \to \underset{\substack{\shortparallel\!\!\!/ \\ 0}}{\pi_3(S^3)} \xrightarrow{\cong} \pi_3(S^2) \to \cancelto{0}{\pi_2(S^1)}$$
$\square$

<u>Rmk</u> Contrast this calculation with the fact that $H_n(S^2) = 0$ for $n > 2$. In fact,

$\pi_n(S^2) \neq 0$ for <u>every</u> $n \geq 2$, although this is far beyond our ability to prove.

<u>Proof of proposition</u> Let $p: E \to B$ be a fiber bundle with fiber F and local trivializations $h_i : p^{-1}(U_i) \xrightarrow{\cong} U_i \times F$, and consider the lifting problem $(D^n \cong [0,1]^n)$

$$\begin{tikzcd}
{[0,1]^n \times \{0\}} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
{[0,1]^{n+1}} \arrow[r, "f"] \arrow[ur, dashed, "\exists ?"] & B.
\end{tikzcd}$$

By compactness, we may subdivide $[0,1]^{n+1}$ into cubes C_j such that $f(C_j)$ lies in some U_i for every j. Constructing the lift inductively over j, we are reduced to the case of the trivial bundle

$$\begin{tikzcd}
{[0,1]^n \times \{0\}} \arrow[r, "\tilde{f}"] \arrow[d] & B \times F \arrow[d] \\
{[0,1]^{n+1}} \arrow[r, "f"] & B
\end{tikzcd},$$

where a lift is given by f in the first coordinate and the composite

$$[0,1]^n \times [0,1] \xrightarrow{\mathrm{id} \times \mathrm{const.}} [0,1]^n \times \{0\} \xrightarrow{\tilde{f}} B \times F \xrightarrow{\mathrm{proj.}} F.$$

$\square$

We still have not calculated a single nonzero higher homotopy group, though we do know they exist.

<u>Goal</u> Calculate $\pi_k S^n$ for $k, n \geq 0$.

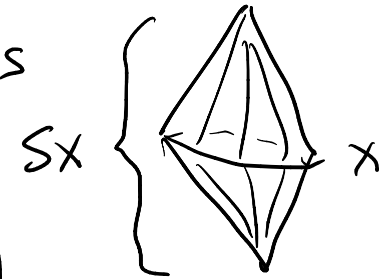
<u>Rmk</u> Most topologists believe that this goal will never be accomplished by human beings.

Our strategy will be to leverage relationships with well-understood objects

<u>Idea 1</u> (Hurewicz) We have maps

$$\pi_k(S^n) \xrightarrow{h_k} H_k(S^n)$$
$$[f: S^k \to S^n] \longmapsto f_*(1).$$

<u>Idea 2</u> (Freudenthal) We have maps



$$\pi_k(S^n) \xrightarrow{S_k} \pi_{k+1}(S^{n+1})$$
$$[f: S^k \to S^n] \longmapsto [Sf: S^{k+1} \to S^{n+1}].$$

<u>Exercise</u> The Hurewicz map h_k is a homomorphism.

<u>Def</u> A space X is <u>n-connected</u> if $\pi_k(X, x_0) = 0$ for $k \leq n$. The <u>connectivity</u> of X is the maximal n such that X is n-connected.

<u>Ex</u> 0-connected $\iff$ path connected

<u>Ex</u> 1-connected $\iff$ simply connected

<u>Warning</u> Do not confuse n-connected with n-connective, a term that we will not use.

<u>Thm</u> (Hurewicz) If X is $(n-1)$-connected for some $n \geq 1$, then

$$h_k : \pi_k(X) \longrightarrow H_k(X)$$

is an isomorphism for every $0 < k \leq n$. In particular, $H_k(X) = 0$ for $0 < k < n$.

<u>Rk</u> If X is 0-connected, then h_1 is the Abelianization homomorphism.

<u>Rk</u> One can prove that h_{n+1} is surjective.

Rk This result, together with our examples from last time, closes the topic of the relationship between homotopy and homology.

Cor For $n>0$, S^n is $(n-1)$-connected, and the homomorphism

$$\begin{aligned} \pi_n(S^n) &\longrightarrow \mathbb{Z} \\ [f] &\longmapsto \deg(f) \end{aligned}$$

is an isomorphism.

Proof Let d be the connectivity of S^n. Since $n>0$, $d\geq 0$. If $d\geq n$, then $H_n(S^n)=0$, a contradiction. If $d<n-1$, then $H_{d+1}(S^n)\neq 0$ a contradiction. Hence $d=n-1$, and the

homomorphism

$$\begin{array}{ccccc} \pi_n(S^n) & \longrightarrow & H_n(S^n) & \longrightarrow & \mathbb{Z} \\ [f] & \longmapsto & f_*(1) & \longmapsto & \deg(f) \end{array}$$

is an isomorphism. $\square$

We turn to the proof.

<u>Observation</u> An n-simplex $\sigma : \Delta^n \to X$ determines an element of $\pi_n(X, x_0)$ if $\sigma(\partial\Delta^n) = \{x_0\}$.

<u>Strategy</u> Define an inverse on the homology arising from such simplices, and show they account for everything.

$$\pi_n(X) \longrightarrow H_n^{(n-1)}(X) \xrightarrow{\cong ?} H_n(X)$$

<u>Notation</u> Write $C_k^{(n)}(X) \subseteq C_k^{(n)}(X)$ for the subgroup generated by simplices $\sigma : \Delta^k \to X$ such that $\sigma|_{\Delta^k_n} \equiv x_0$. Write $H_*^{(n)}(X)$ for the homology of the subcomplex $C_*^{(n)}(X)$.

<u>Lemma</u> If X is n-connected, then the inclusion $C_*^{(n)}(X) \subseteq C_*(X)$ is a chain homotopy equivalence.

<u>Proof</u> Given $\sigma : \Delta^k \to X$, we construct a homotopy $H_\sigma : \Delta^k \times [0,1] \to X$ such that, (1) $H_\sigma(-,1) \in C_k^{(n)}(X)$, (2) $H_\sigma(-,t) = \sigma$ if $\sigma \in C_k^{(n)}(X)$, and (3) the diagrams shown at right all commute:

$$\begin{tikzcd} \Delta^{k-1} \times [0,1] \arrow[r, "\eta_i \times id"] \arrow[dr, bend right, "H_{\sigma \circ \eta_i}"'] & \Delta^k \times [0,1] \arrow[d, "H_\sigma"] \\ & X \end{tikzcd}$$

We proceed by induction on k, the base case $k=0$ being supplied by choosing a path to x_0. For the first induction step, take $k \leq n$. The maps $H_{\sigma \circ \eta_i}$ and σ determine the map f in the diagram

$$
\begin{tikzcd}
\Delta^k \times [0,1] \arrow[d, two heads] & \supseteq & \partial\Delta^k \times [0,1] \cup \Delta^k \times \{0\} \arrow[r, "f"] \arrow[d, two heads] & X \\
\dfrac{\Delta^k \times [0,1]}{\Delta^k \times \{1\}} \arrow[d, phantom, "\cong" rotate=-90] & \supseteq & \dfrac{\partial\Delta^k \times [0,1] \cup \Delta^k \times \{0\}}{\partial\Delta^k \times \{1\}} \arrow[ur, dashed, "\tilde{f}"] \arrow[d, phantom, "\cong" rotate=-90] & \\
\Delta^{k+1} & & \partial\Delta^{k+1} \arrow[ll, hook'] &
\end{tikzcd}
$$

Since $k \leq n$, $f|_{\partial\Delta^k \times \{1\}} \equiv x_0$, so the dashed filler exists. Since

$\pi_k(X)=0$ by assumption, $\tilde{f}$ extends over Δ^{k+1}, and we define H_σ to be the resulting composite

$$H_\sigma : \Delta^k \times [0,1] \twoheadrightarrow \Delta^{k+1} \longrightarrow X.$$

The three desired properties are immediate from the construction.

For the second induction step, take $k>n$, and define H_σ to be any dashed filler in

$$\begin{tikzcd} \partial\Delta^k \times [0,1] \cup \Delta^k \times \{0\} \arrow[r] \arrow[d, hook, "i"'] & X \\ \Delta^k \times [0,1] \arrow[ur, dashed, "H_\sigma"'] & \end{tikzcd}$$

Such a filler exists because the map $X \longrightarrow pt$ is a fibration.

Next, define $\rho : C_k(X) \longrightarrow C_k^{(n)}(X)$ by $\rho(\sigma) = H_\sigma(-,1)$ and $\Gamma : C_k(X) \longrightarrow C_{k+1}(X)$ by

$$\Gamma\sigma = \sum_{i=1}^{k+1} (-1)^i H_\sigma \circ P_i .$$

$$\begin{tikzcd} \Delta^{k+1} \arrow[d, "P_i"] \\ \Delta^k \times [0,1] \arrow[d, "H_\sigma"] \\ X \end{tikzcd}$$

The properties of H_σ imply (exercise) that $\partial\Gamma + \Gamma\partial = \iota \circ \rho - \mathrm{id}$, where ι is the inclusion of $C_k^{(n)}(X)$.

$\square$