PROBLEM LIST 1
TOPOLOGY 3, SPRING 2024

Problem 1. Without using the Seifert–van Kampen theorem, decide whether the fundamental group of each of the following spaces is trivial, infinite cyclic, or non-Abelian. Justify your answer.

(1) The solid torus $D^2 \times S^1$.
(2) The punctured torus $T^2 \setminus \{(1,1)\}$.
(3) The cylinder $S^1 \times [0,1]$.
(4) The twice-punctured plane $\mathbb{R}^2 \setminus \{(0,1),(0,-1)\}$.
(5) The space of vectors in $\mathbb{R}^2$ of length strictly greater than 1.
(6) The space of vectors in $\mathbb{R}^2$ of length at least 1.
(7) The space of vectors in $\mathbb{R}^2$ of length strictly less than 1.
(8) The union in $\mathbb{R}^2$ of the unit circle and the positive x-axis.
(9) The union in $\mathbb{R}^2$ of the unit circle and the half-plane $\{(x,y) : x \geq 0\}$.
(10) The union in $\mathbb{R}^2$ of the unit circle and the x-axis.

This problem is mostly Munkres 9.58.2, although our notation differs.

Problem 2. In this problem, you will show that Abelian fundamental groups at different base points are *canonically* isomorphic. More specifically, given points x_0 and x_1 in the path connected space X, show that $\pi_1(X, x_0)$ is Abelian if and only if $\hat{\alpha} = \hat{\beta}$ for every pair of paths α and β from x_0 to x_1. This problem is Munkres 9.52.3.

Problem 3. Let G_1 and G_2 be nontrivial groups, and define $G = G_1 * G_2$.

(1) Show that G is non-Abelian.

Given $g \in G$, define the *length* of g, denoted $\ell(g)$, to be the length of the unique reduced word in G_1 and G_2 representing g.

(2) Show that, if $\ell(g) > 0$ is even, then g does not have finite order.
(3) Show that, if $\ell(g) > 1$ is odd, then g is conjugate to an element g' such that $\ell(g') < \ell(g)$.
(4) Show that, if $g \in G$ has finite order, then g is conjugate to an element of G_1 or an element of G_2. When does the converse hold?

This problem is Munkres 68.2.

Problem 4. Let X be a space, $x_0 \in X$ a point, and $X_i \subseteq X$ a closed subspace for $1 \leq i \leq r$. If $X = \bigcup_{i=1}^r X_i$ and $X_i \cap X_j = \{x_0\}$ for $i \neq j$, then we say that X is the *wedge* of the spaces X_i and write $X = X_1 \vee \cdots \vee X_r$.

(1) Suppose that x_0 is a deformation retract of an open subset W_i of X_i for each $1 \leq i \leq r$. Show that the inclusions of the X_i induce an isomorphism

$$\pi_1(X_1, x_0) * \cdots * \pi_1(X_r, x_0) \xrightarrow{\simeq} \pi_1(X, x_0).$$

(2) Fix $n_i \geq 0$ for $1 \leq i \leq r$. Calculate $\pi_1(S^{n_1} \vee \cdots \vee S^{n_r}, x_0)$.

This problem includes Munkres 70.2 and 70.3.

Problem 5. Let GL_2 denote the set of invertible 2×2 matrices with real entries. This set inherits a topology as a subspace of $\mathbb{R}^4$ (the space of *all* 2×2 matrices with real entries).

(1) Show that matrix multiplication $m : GL_2 \times GL_2 \to GL_2$ and matrix inversion $i : GL_2 \to GL_2$ are continuous (you may assume that addition and multiplication of real numbers are continuous).
(2) Show that the operation $\otimes$ defined by $[\gamma_1] \otimes [\gamma_2] = [m(\gamma_1, \gamma_2)]$ defines a group law on $\pi_1(GL_2, I)$, where I is the identity matrix.
(3) Show that the group law defined in (2) is the same as the usual group law defined by concatenation of loops.
(4) Use (2) to show that $\pi_1(GL_2, I)$ is Abelian.

This problem is valid for matrices of any size; in fact, it is valid for any *topological group* (see Munkres 9.52.7).

Problem 6. Prove the compression criterion: the element $[f] \in \pi_n(X, A, x_0)$ is trivial if and only if there is a map $\tilde{f} : D^n \to A$ making the diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & A \\ \downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow \\ D^n & \xrightarrow{f} & X \end{array}$$

commute up to homotopy rel S^{n-1}. Beware spoilers everywhere.

Problem 7. Let Σ_g denote the compact, orientable surface of genus g.

(1) Use the Van Kampen theorem to calculate $\pi_1(\Sigma_g)$.
(2) Show that $\pi_n(\Sigma_g) = 0$ for $g > 0$ and $n > 1$.
(3) What about non-orientable surfaces?

Beware spoilers everywhere.

Problem 8. Show that S^∞ is contractible. Beware spoilers in Hatcher.