Last time

- Whitney + Sard
- $\pi_k(X, X_n) = 0$ for $k \leq n$
- Existence and uniqueness of $K(G,n)$.

Goal Convert maps into fibrations.

Q What kind of fibrations are there?

Q What kind of covering spaces are there?

As a starting point, let's ask when a space has a simply connected cover (like $\mathbb{R} \to S^1$).

We begin by observing that some assumption is necessary.

<u>Lemma</u> Let $p: E \to B$ be a covering map, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. If E is simply connected, then there is an open subset $b_0 \in U \subseteq B$ such that the homomorphism $\pi_1(U, b_0) \xrightarrow{i_*} \pi_1(B, b_0)$ induced by the inclusion is trivial.

<u>Proof</u> Take U to be evenly covered by p. Choose $e_0 \in p^{-1}(b_0)$ and let $V \ni e_0$ be such that $p|_V$ is a homeomorphism onto U. The claim follows by considering the commutative diagram

$$\begin{array}{ccc} \pi_1(V,e_0) & \longrightarrow & \pi_1(E,e_0) \nearrow 0 \\ p_* \big\downarrow \cong & & \big\downarrow p_* \\ \pi_1(U,b_0) & \longrightarrow & \pi_1(B,b_0). \end{array}$$

□

__Def__ We say that a space B is __semilocally simply connected__ if the conclusion of the lemma holds for every $b_0 \in B$.

__Ex__ Let X be the Hawaiian earring. Any neighbourhood U of the origin in X contains the circle C_n of radius $1/n$ for some $n \gg 0$. In the diagram of homomorphisms

$$\begin{array}{ccc} \pi_1(C_n) & \longrightarrow & \pi_1(X) \\ & \searrow \quad \nearrow & \\ & \pi_1(\mathcal{U}) & \end{array}$$

the top map is injective, since C_n is a retract of X, and $\pi_1(C_n)$ is non-trivial, so the righthand map cannot be trivial. Thus, X is not semilocally simply connected and so has no simply connected covering space.

<u>Thm</u> If B is path connected, locally path connected, and semilocally simply connected, then B admits a simply connected covering space.

This covering space is unique in an appropriate sense and called the <u>universal cover</u> of B.

<u>Observation</u> If $p: E \to B$ is a covering map with E simply connected, then

$$\left\{ \begin{array}{c} \text{points of} \\ E \end{array} \right\} \underset{\substack{\uparrow \\ \text{simply connected}}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{path homotopy classes} \\ \text{of paths starting at } e_0 \end{array} \right\} \underset{\substack{\uparrow \\ \text{lifting}}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{homotopy classes} \\ \text{of paths starting} \\ \text{at } b_0 \end{array} \right\}.$$

So define $E = \{ [\gamma] : \gamma : [0,1] \to B, \ \gamma(0) = b_0 \}$ and $p: E \to B$ by $p([\gamma]) = \gamma(1)$. How to topologize E?

Lemma 1 Let $\mathcal{U} \subseteq B$ be a path connected open subset such that $\pi_1(\mathcal{U}) \to \pi_1(B)$ is trivial for some (hence any) basepoint. Choose $\gamma : [0,1] \to B$ with $\gamma(1) \in \mathcal{U}$, and set

$$\mathcal{U}_{[\gamma]} = \{ [\gamma \star \delta] : \delta : [0,1] \to \mathcal{U},\ \delta(0) = \gamma(1) \} \subseteq E.$$

Then p maps $\mathcal{U}_{[\gamma]}$ bijectively onto $\mathcal{U}$.

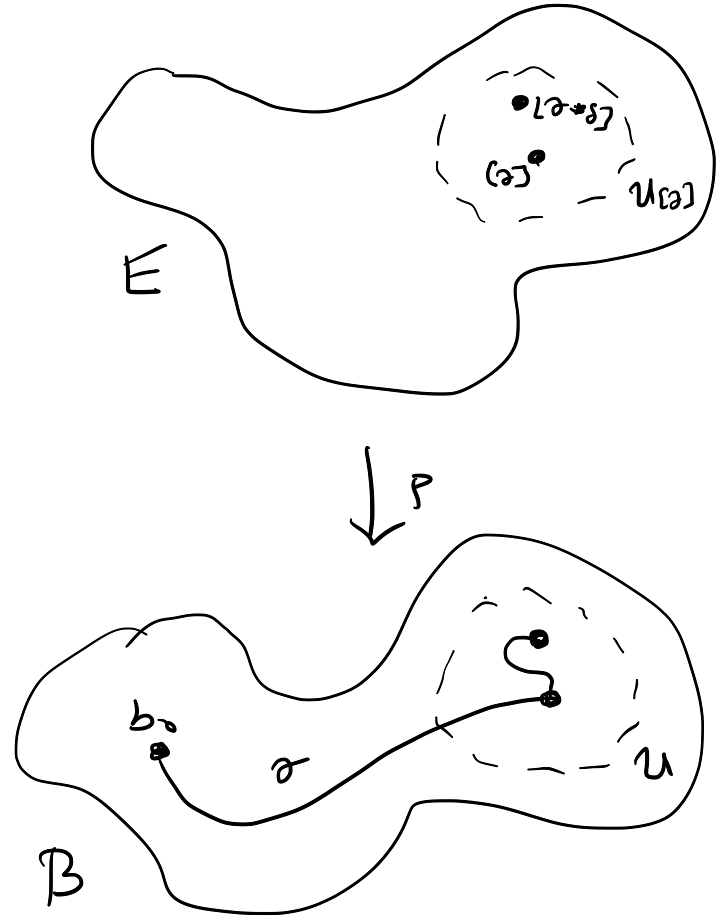
Proof Surjectivity follows from the assumption that $\mathcal{U}$ is path connected. If δ' is a path in $\mathcal{U}$ such that $\delta'(0) = \gamma(1)$ and $\delta'(1) = \delta(1)$, then δ and δ' are path homotopic in B by assumption, so $[\gamma * \delta] = [\gamma * \delta']$, which shows injectivity.

$\square$

Let $\mathcal{B}$ denote the set of open subsets $U \subseteq B$ such that U is path connected and $\pi_1(U) \to \pi_1(B)$ is trivial.

Since B is semilocally simply connected, $\mathcal{B}$ is a basis for the topology of B (exercise). Define

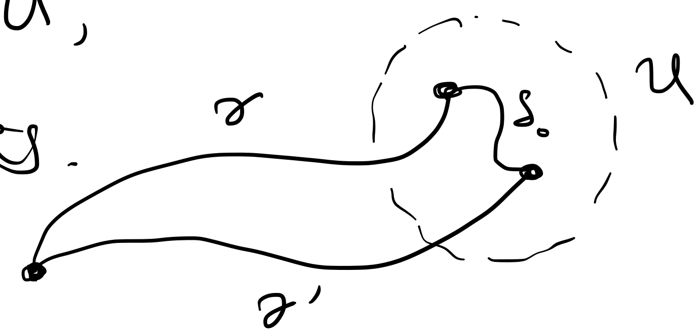
$$\tilde{\mathcal{B}} = \{ U_{[\gamma]} : U \in \mathcal{B},\ \gamma(1) \in U \}.$$



<u>Lemma 2</u> $\tilde{\mathcal{B}}$ is a basis.

Lemma 3 If $[\gamma'] \in U_{[\gamma]}$, then $U_{[\gamma']} = U_{[\gamma]}$.

Proof By assumption, $[\gamma'] = [\gamma * \delta_0]$ for some $\delta_0 : [0,1] \to U$. Given $\delta : [0,1] \to U$, consider the following possibilities.



If $\delta(0) = \gamma(1)$, then

$$[\gamma * \delta] = [\gamma' * (\bar{\delta}_0 * \delta)] \in U_{[\gamma']}.$$

On the other hand, if $\delta(0) = \gamma'(1)$, then

$$[\gamma' * \delta] = [\gamma * (\delta_0 * \delta)] \in U_{[\gamma]}.$$

$\square$

Proof of Lemma 2 Given $[\gamma] \in E$, $\gamma(1) \in U$ for some $U \in \mathcal{B}$, and $[\gamma] \in U_{[\gamma]}$, so (B1) holds.

For (B2), suppose $U_{[\gamma]} \cap V_{[\gamma']} \ni [\gamma'']$. Then $\gamma''(1) \in U \cap V$, and we may choose $W \in \mathcal{B}$ s.t. $\gamma''(1) \in W \subseteq U \cap V$. By Lemma 3, $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$, and so

$$[\gamma''] \in W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']} = U_{[\gamma]} \cap V_{[\gamma']}.$$

$\square$

We equip $\tilde{E}$ with the topology generated by the basis $\tilde{\mathcal{B}}$.

<u>Lemma 4</u> For every $U_{[\gamma]} \in \tilde{\mathcal{B}}$, p maps $U_{[\gamma]}$ homeomorphically onto U.

Proof It suffices by Lemma 1 to show that $p|_{\mathcal{U}_{[\gamma]}}$ is continuous and open. For the first claim, given $V \subseteq U$ with $V \in \mathcal{B}$, let $[\gamma'] \in \mathcal{U}_{[\gamma]}$ be such that $\gamma'(1) \in V$. Then

$$p^{-1}(V) \cap \mathcal{U}_{[\gamma]} = p^{-1}(V) \cap \mathcal{U}_{[\gamma']} \supseteq V_{[\gamma']} \ni [\gamma'],$$

as desired. For the second claim, $p(V_{[\gamma]}) = V$ is open.

$\square$

Corollary The map $p : E \to B$ is a covering map.

Proof Continuity is a local property, $p|_{\mathcal{U}_{[\gamma]}}$ is continuous for $\mathcal{U}_{[\gamma]} \in \tilde{\mathcal{B}}$ by Lemma 4. Given

$b \in B$, choose $b \in \mathcal{U} \in \mathcal{B}$. We claim that $\mathcal{U}$ is evenly covered by p. Indeed,

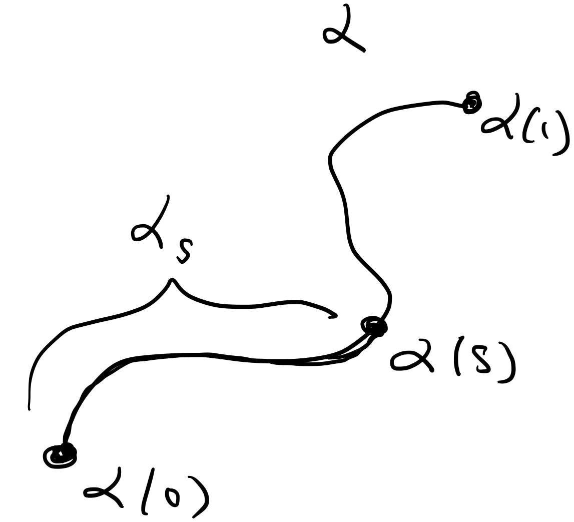
$$p^{-1}(\mathcal{U}) = \bigcup_{[\gamma] : \gamma(1) \in \mathcal{U}} \mathcal{U}_{[\gamma]},$$

and the $\mathcal{U}_{[\gamma]}$ are either disjoint or equal by Lemma 3. The claim now follows from Lemma 4. $\square$

In order to show that E is simply connected, it will be convenient to know how paths lift along p. Let $e_0 \in E$ be the point given by the homotopy class of the constant path at b_0. Given a path $\alpha : [0,1] \to B$ with

$\alpha(0) = b_0$, define $\tilde{\alpha} : [0,1] \to E$ by

$$\tilde{\alpha}(s) = [\alpha_s],$$



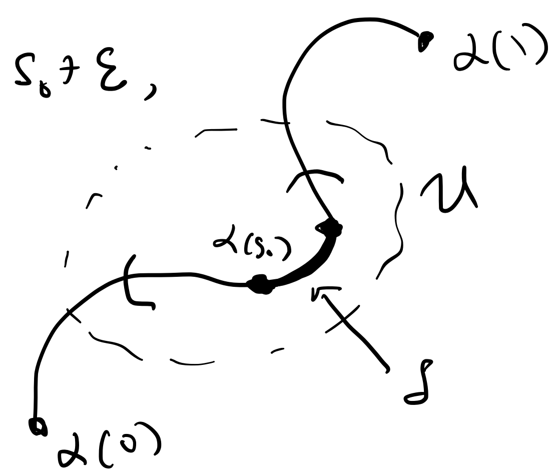
where $\alpha_s(t) = \alpha(st)$. Note that $(p \circ \tilde{\alpha})(s) = \alpha_s(1) = \alpha(s)$, so the notation is justified once we verify the following.

<u>Lemma 5</u> $\tilde{\alpha}$ is continuous.

<u>Proof of theorem</u> Given $[\gamma] \in E$, $\tilde{\gamma}$ is a path from e_0 to $[\gamma]$, so E is path connected. It remains to check that $p_*(\pi_1(E, e_0))$ is trivial. The subgroup $p_*(\pi_1(E, e_0))$ consists

of classes of loops γ such that $\tilde{\gamma}$ is also a loop, i.e., $\tilde{\gamma}(1) = e_0$. But $\tilde{\gamma}(1) = [\gamma]$, so $[\gamma]$ is equal to the class of the constant loop at b_0, as desired. $\square$

<u>Proof of Lemma 5</u> Given $s_0 \in [0,1]$ and $U \in \mathcal{B}$ with $\alpha(s_0) \in U$, there exists $\varepsilon > 0$ such that $\alpha((s_0 - \varepsilon, s_0 + \varepsilon)) \subseteq U$. We will show that $\tilde{\alpha}((s_0 - \varepsilon, s_0 + \varepsilon)) \subseteq U_{[\alpha_{s_0}]}$. Given $s_0 \leq s_1 < s_0 + \varepsilon$, let $\delta(s) = \alpha(s_0 + s(s_1 - s_0))$.



Then

$$\tilde{\alpha}(s_1) = [\alpha_{s_1}] = [\alpha_{s_0} * \delta] \in U_{[\alpha_{s_0}]},$$

since the image of δ lies in U. The argument for $s_0 - \varepsilon < s_1 \leq s_0$ is similar.

$\square$

What other covering spaces are there? The following is a clue.

<u>Prop</u> Let $p: E \to B$ be a covering map, $b_0 \in B$ and $e_0, e_1 \in p^{-1}(b_0)$.

(1) $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.

(2) If e_0 and e_1 lie in the same path component of E, then $p_*(\pi_1(E, e_0))$ and $p_*(\pi_1(E, e_1))$ are conjugate subgroups of $\pi_1(B, b_0)$.

Proof The first claim is immediate from the LES in homotopy for p, since $\pi_1(F, e_0)$ is trivial. For the second, given a path $\tilde{\alpha}$ from e_0 to e_1, set $\alpha = p \circ \tilde{\alpha}$, and consider the commuting diagram

$$\begin{array}{ccc} \pi_1(E, e_0) & \xrightarrow[\cong]{\hat{\tilde{\alpha}}} & \pi_1(E, e_1) \\ p_* \downarrow & & \downarrow p_* \\ \pi_1(B, b_0) & \xrightarrow{\hat{\alpha}} & \pi_1(B, b_0). \end{array}$$

Since α is a loop at b_0, $\hat{\alpha}([\gamma]) = [\alpha]^{-1}[\gamma][\alpha]$.

$\square$

So $p \longmapsto p_*(\pi_1(E, e_0))$ determines assignments

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{path connected}\\ \text{covering spaces of } B\\ \text{w/ choice of basepoint}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{subgroups of}\\ \pi_1(B, b_0)\end{array}\right\} \\ \downarrow & & \downarrow \\ \left\{\begin{array}{c}\text{path connected}\\ \text{covering spaces of } B\end{array}\right\} & \dashrightarrow & \left\{\begin{array}{c}\text{subgroups of}\\ \pi_1(B, b_0)\end{array}\right\} \Big/ \text{conjugacy} \end{array}$$

<u>Revised question</u> Is this assignment surjective?

If so, what is the induced equivalence relation on the source?

<u>Clue</u> By the orbit stabilizer theorem, if X is a transitive G-set, a choice of $x \in X$ gives a bijection $G/G_x \xrightarrow{\cong} X$.

So we have bijections

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{transitive } G\text{-sets} \\ \text{w/ choice of element} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{subgroups} \\ \text{of } G \end{array} \right\} \\ \downarrow & & \downarrow \\ \left\{ \begin{array}{c} \text{transitive} \\ G\text{-sets} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{subgroups} \\ \text{of } G \end{array} \right\} / \text{conjugacy} \end{array}$$