Last time

- What covering spaces are there?
- Universal cover

$$p : E \to B \longmapsto p_*(\pi_1(E))$$

$$\begin{tikzcd}
\left\{ \begin{matrix} \text{path connected} \\ \text{covering spaces} \end{matrix} \right\} \arrow[rr] & & \left\{ \begin{matrix} \text{subgroups} \\ \text{of } \pi_1(B, b_0) \end{matrix} \right\} \Big/ \text{conjugacy} \arrow[d, shift right] \arrow[d, phantom, "\cong"'] \\
& & \left\{ \begin{matrix} \text{transitive} \\ G\text{-sets} \end{matrix} \right\} \Big/ \text{isomorphism} \arrow[u, shift right] \arrow[ull, dashed, "\exists ?"]
\end{tikzcd}$$

$$H \longmapsto G/H \qquad \qquad G_x \longmapsfrom X$$

Notation For us, G/H denotes the set of **right** cosets of H in G, i.e., $G/H = \{Hg : g \in G\}$, where

$$Hg = \{hg : h \in H\}.$$

Elements g_1 & g_2 lie in the same coset if and only if $g_1 g_2^{-1} \in H$.

Q What do covering spaces have to do with group actions?

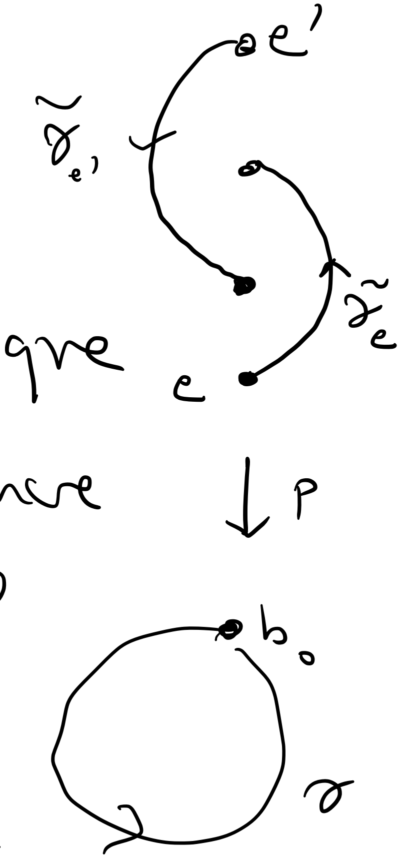
Set $G := \pi_1(B, b_0)$, and define

$$p^{-1}(b_0) \times G \longrightarrow p^{-1}(b_0)$$

$$(e, [\gamma]) \longmapsto e \cdot [\gamma] := \tilde{\gamma}_e(1)$$

where $\tilde{\gamma}_e$ is the unique lift of γ with $\tilde{\gamma}_e(0) = e$.

<u>Lemma</u> The function $(e,[\gamma]) \mapsto e\cdot[\gamma]$ is a G-action.



<u>Proof</u> First, it is well-defined by unique lifting of path homotopies. Second, since the unique lift of the constant loop is constant for every $e \in p^{-1}(b_0)$, we have $e \cdot 1 = e$. Finally, note that $(e\cdot[\gamma])\cdot[\gamma']$ is the endpoint of the unique lift of γ' starting at $\tilde{\gamma}_e(1)$, while $e\cdot([\gamma]\cdot[\gamma'])$ is the endpoint of the unique lift of $\gamma \star \gamma'$ starting at e, and these coincide.

$\square$

Since the stabilizer of any $e \in \bar{p}^{-1}(b_0)$ is $H := p_*(\pi_1(E, e))$ (exercise), the bijection of the lifting correspondence is simply orbit-stabilizer applied to the G-set $\bar{p}^{-1}(b_0)$. In particular, it is an isomorphism <u>of G-sets</u>.

<u>Idea</u> The fiber of the universal cover is the G-set G, and the fiber of the (hypothetical) covering space corresponding to H is G/H. So construct the latter from the former by "quotienting by H".

<u>Question</u> How does G act on the universal <u>cover?</u>

If we want to extend the action on the fiber, then $[\gamma] \in \pi_1(B, b_0)$ with $\tilde{\gamma}(0) = e_0$ and $\tilde{\gamma}(1) = e$ should produce the filler

$$\begin{tikzcd} pt \arrow[r, "e_1"] \arrow[d, "e_0"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] \arrow[ur, dashed, "\exists ?"] & B \end{tikzcd}$$

This requires a more general lifting property.

Lemma Given $f: Y \to B$ with Y (locally) path connected and $y_0 \in f^{-1}(b_0)$, there is a unique lift $\tilde{f}$ of f along p with $\tilde{f}(y_0) = e_0$ iff $\mathrm{im}(f_*) \subseteq \mathrm{im}(p_*)$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Def Let $p_i: E_i \to B$ be covering maps. A map $f: E_1 \to E_2$ is called an equivalence (of covering spaces) if f is a homeomorphism and $p_2 \circ f = p_1$.

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Assumption B path connected and locally path connected.

Thm Let $p: E \to B$ and $p': E' \to B$ be covering with E & E' path connected, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, and $e_0' \in (p')^{-1}(b_0)$.
There is an equivalence $f: E \to E'$ of covering spaces such that $f(e_0) = e_0'$ if and only if

$$H := p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e_0')) =: H'.$$

If so, this equivalence is unique.

Cor $\pi_1(B, b_0)$ acts by equivalences on the universal cover of B.

Cor The dashed arrows below exist and are injective.

$$
\begin{tikzcd}
\{\text{— " —}\}/\text{pointed equivalence} \arrow[dr, dashed] & \\
\{\text{path connected pointed covers}\} \arrow[u] \arrow[r] \arrow[d] & \{\text{subgroups of } \pi_1(B, b_0)\} \arrow[d] \\
\{\text{path connected covers}\} \arrow[r] \arrow[d] & \{\text{subgroups of } \pi_1(B, b_0)\}/\text{conjugacy} \\
\{\text{— " —}\}/\text{equivalence} \arrow[ur, dashed] &
\end{tikzcd}
$$

<u>Proof</u> The claim about the top arrow and the existence of the bottom are essentially immediate. For injectivity, set $H = p_*(\pi_1(E, e_0))$ and $H' = p_*(\pi_1(E', e_0'))$, and suppose that $H' = [\gamma]^{-1} H [\gamma]$. Define $e_0'' := \tilde{\gamma}(1) \in (p')^{-1}(b_0)$. From the diagram

$$\begin{tikzcd}
\pi_1(E', e_0') \arrow[r, "\widehat{\gamma}"] \arrow[d, "p'_*"'] & \pi_1(E', e_0'') \arrow[d, "p'_*"] \\
\pi_1(B, b_0) \arrow[r, "\hat{\gamma}"] & \pi_1(B, b_0)
\end{tikzcd}$$

we have $H = p_*(\pi_1(E', e_0''))$, so the theorem applies. $\square$

__Proof of theorem__ Let $p: E \to B$ and $p': E' \to B$ be covering maps with E and E' path connected. Since B is locally path connected, so are E and E' (p & p' are local homeomorphisms).

The "only if" direction is clear, so choose b_0, e_0, and e_0', and assume that

$$H := p_*(\pi_1(E, e_0)) = p'_*(\pi_1(E', e_0)) =: H'.$$

Applying the lemma twice, we obtain the unique lifts

$$\begin{tikzcd} & E' \arrow[d, "p'"] \\ E \arrow[ur, dashed, "\tilde{p}"] \arrow[r, "p"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & E \arrow[d, "p"] \\ E' \arrow[ur, dashed, "\tilde{p}'"] \arrow[r, "p'"'] & B \end{tikzcd} .$$

From the diagram

$$\begin{tikzcd}
E \arrow[r, "\tilde{p}'"] \arrow[dr, two heads, "p"'] & E' \arrow[r, "\tilde{p}"] \arrow[d, two heads, "p'"] & E \arrow[dl, two heads, "p"] \\
& B &
\end{tikzcd},$$

we see that $\tilde{p} \circ \tilde{p}'$ is a lift of p along p, hence equal to the identity by uniqueness. Similarly, $\tilde{p}' \circ \tilde{p} = \mathrm{id}_{E'}$, so $\tilde{p}$ is the desired equivalence. $\square$

It remains to prove the lemma.

<u>Idea</u> We already know how to lift paths.

<u>Proof</u> Since $f_* = p_* \circ \tilde{f}_*$, the condition

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(E, e_0))$$

is necessary. Moreover, if $\tilde{f}$ exists, then, given $y \in Y$, there is a path $\gamma_y : [0,1] \to Y$ from y_0 to y by path connectedness, and $\tilde{f} \circ \gamma_y = \widetilde{f \circ \gamma_y}$ by unique path lifting. Therefore,

$$\tilde{f}(y) = \widetilde{f \circ \gamma_y}(1)$$

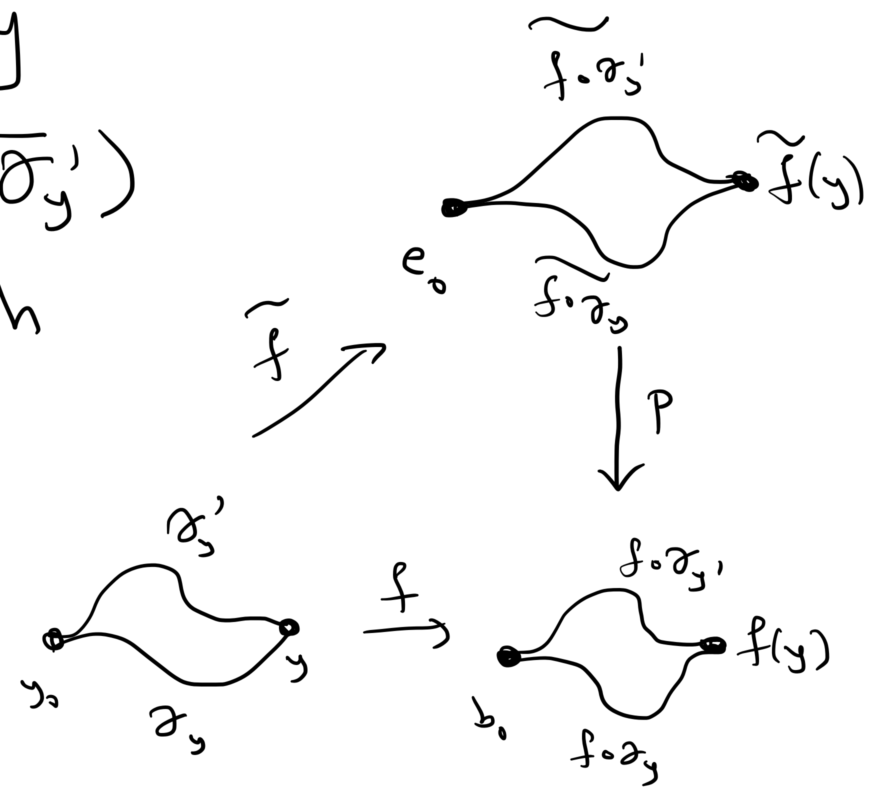
which shows uniqueness and supplies a candidate definition. It remains to show that $\tilde{f}$, so defined, is independent of γ_y

and continuous.

If γ_y' is another path from y_0 to y, then our assumption on $f_*(\pi_1(Y, y_0))$ implies that there is a loop $\delta : [0,1] \to E$ based at e_0 such that

$$[p \circ \delta] = [f \circ (\gamma_y * \overline{\gamma_y'})].$$

Thus, there is a path homotopy $H : [0,1] \times [0,1] \to B$ from $f \circ (\gamma_y * \overline{\gamma_y'})$ to $p \circ \delta$, which lifts to a path homotopy $\tilde{H} : [0,1] \times [0,1] \to E$.



Now, since $\tilde{H}(-,1) = \delta$ is a loop at e_0, so is $\tilde{H}(-,0)$. But $\tilde{H}(-,0)$ is the concatenation of $\widetilde{f\circ\gamma_y}$ and the unique lift of $f\circ\overline{\gamma_y'}$ starting at $\widetilde{f\circ\gamma_y}(1)$. Hence this latter lift ends at e_0 and is therefore the reverse of $\widetilde{f\circ\gamma_y'}$. Thus,

$$\widetilde{f\circ\gamma_y}(1) = \widetilde{f\circ\gamma_y'}(1),$$

so $\tilde{f}$ is well-defined.

For continuity, given $y_1 \in Y$ and an open set $\tilde{f}(y_1) \in V \subseteq E$, we wish to find an open set $y_1 \in W \subseteq Y$ such that $\tilde{f}(W) \subseteq V$. Since p is a covering map, we may assume that $p|_V$ is a homeomorphism onto $U := p(V)$. Fix a path $\gamma_0 : [0,1] \to Y$ from y_0 to y_1 and a path connected open subset

$y_1 \in W \subseteq f^{-1}(\mathcal{U})$ (here we use local path connectivity). Then, for $y \in W$, we may compute $\tilde{f}(y)$ as the endpoint of $\widetilde{f \circ \gamma_y}$, where γ_y is any path from y_0 to y. We take γ_y to be the path obtained by concatenating γ_0 and a path δ_y in W from y_1 to y, which exists by our assumption on W. Then $\tilde{f}(y)$ is the endpoint of the lift of $f \circ \delta_y$ starting at y_1, which lies in V, since the image of δ_y lies in $W \subseteq f^{-1}(\mathcal{U})$. $\square$