Last time

- $\pi_1(B)$ acts on the universal cover
- Equivalences of covering spaces
- $(E, e_0) \cong (E', e_0') \iff H = H'$
- General lifting lemma

$$\begin{tikzcd} pt \arrow[r, "e_0"] \arrow[d, "y_0"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"] \arrow[ur, dashed] & B \end{tikzcd}$$

As a consequence of our work last time, the function

$$\left\{ \begin{array}{c} \text{path connected} \\ \text{covers of } B \end{array} \right\} \Big/ \text{equivalence} \xrightarrow{\; p \longmapsto p_*(\pi_1(E)) \;} \left\{ \begin{array}{c} \text{subgroups} \\ \text{of } \pi_1(B, b_0) \end{array} \right\} \Big/ \text{conjugacy}$$

is well-defined and injective. Is it surjective?

Let us return to the action of $G := \pi_1(B, b_0)$ on the universal cover $\tilde{B}$, which works as follows. Given $[\gamma] \in G$, there is the unique lift $f_{[\gamma]}$ in the diagram

$$\begin{tikzcd} pt \arrow[r, "\tilde{\gamma}(1)"] \arrow[d, "e_0"'] & \tilde{B} \arrow[d] \\ \tilde{B} \arrow[r] \arrow[ur, dashed] & B, \end{tikzcd}$$

where $e_0 \in p^{-1}(b_0)$ is fixed. It is an equivalence since $f_{[\gamma]} \circ f_{[\gamma]^{-1}} = id$ by uniqueness, and vice versa

Now, given $[\delta] \in G$,

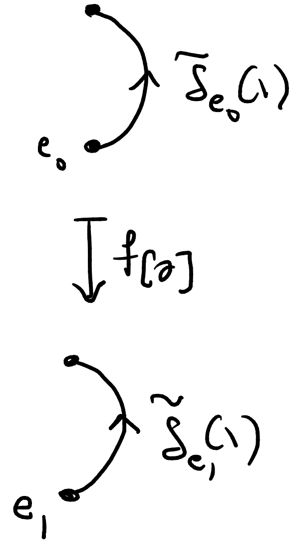
$$f_{[\gamma]} f_{[\delta]}(e_0) = f_{[\gamma]}(e) \qquad e := \tilde{\delta}_{e_0}(1)$$

$$= \tilde{\delta}_{e_1}(1) \qquad e_1 := \tilde{\gamma}_{e_0}(1)$$

$$= \tilde{\gamma}_{e_0} \star \tilde{\delta}_{e_1}(1)$$

$$= \widetilde{(\gamma \star \delta)}_{e_0}(1)$$

$$= f_{[\gamma][\delta]}(e_0),$$



hence $f_{[\gamma]} f_{[\delta]} = f_{[\gamma][\delta]}$ by uniqueness. So we have defined a G-action — but a <u>left</u> G-action! We therefore have two actions of G on $p^{-1}(b_0)$, one left and one right. Under the bijection $G \cong p^{-1}(b_0)$ given by the choice of e_0, these actions correspond to the respective actions of G on itself by left and right multiplication.

Definition A (left) G-*space* is a G-set X with a topology such that the function

$$\begin{aligned} X &\longrightarrow X \\ x &\longmapsto gx \end{aligned}$$

is continuous for every $g \in G$

Ex $\tilde{B}$ is a left G-space hence a left H-space for any $H \leq G$.

Definition Let X be a left G-space. The **orbit space** of X is the set

$$X/G = \{ Gx : x \in X \}$$

of orbits, equipped with quotient topology from the projection $X \xrightarrow{q} X/G$.

Exercise If $p: E \to B$ is a covering map and E a G-space such that $e \mapsto g \cdot e$ is an equivalence of covering spaces for every $g \in G$, then the induced map $\bar{p}: E/G \to B$ is a covering map.

Prop Fix $e_0 \in \bar{p}^{-1}(b_0)$, so that G acts on $\tilde{B}$. For any $H \leq G$, $\bar{p} : \tilde{B}/H \to B$ is a path connected covering space with $\bar{p}_*(\pi_1(\tilde{B}/H, He_0)) = H$.

Proof Since $\tilde{B}/H$ is the continuous image of $\tilde{B}$, it is path connected, and $\bar{p}$ is a covering map by the exercise. Now, $\bar{p}_*(\pi_1(\tilde{B}/H, He_0))$ is the subgroup of $[\gamma]$ such that the lift $\tilde{\gamma}$ along $\bar{p}$ starting at He_0 is a loop. But $\tilde{\gamma} = q \circ \tilde{\tilde{\gamma}}$, where $\tilde{\tilde{\gamma}}$ is the lift of γ along p, so $\tilde{\gamma}$ is a loop if and only if $\exists [\delta] \in H$ such that

$$\begin{tikzcd}
 & \tilde{B} \arrow[d, "q"] \\
 & \tilde{B}/H \arrow[d, "\bar{p}"] \\
{[0,1]} \arrow[r, "\gamma"] \arrow[ruu, dashed, "\tilde{\tilde{\gamma}}"] \arrow[ru, dashed, "\tilde{\gamma}"] & B
\end{tikzcd}$$

$$\tilde{\tilde{\gamma}}(1) = [\delta]\cdot e_0$$
$$= f_{[\delta]}(e_0)$$
$$= \tilde{\tilde{\delta}}(1) \iff [\gamma]=[\delta],$$

since $\tilde{B}$ is simply connected. Thus, $\tilde{\gamma}$ is a loop if and only if $[\gamma]\in H$. $\square$

One can go further and classify automorphisms of covering spaces (so-called "deck transformations"). All of this may be summarized by the Galois correspondence:

$H \leq G \longleftrightarrow$ Cover $p: E \to B$ with E path connected

$[G:H] \longleftrightarrow$ degree of p

$N(H)/H \longleftrightarrow$ deck transformations

$H \trianglelefteq G \longleftrightarrow$ deck transformations act transitively on fibers

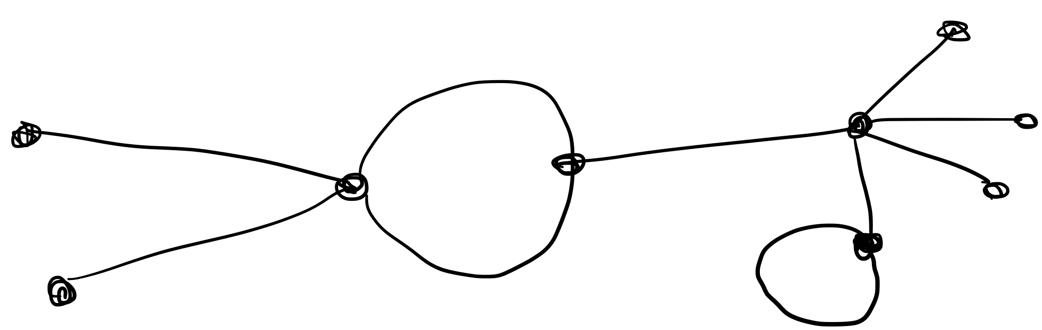
<u>Rmk</u> In algebraic geometry, Galois theory and covering space theory are the same.

<u>Rmk</u> This can be upgraded to an equivalence of categories between covering spaces and G-sets.

We close by sketching an application of covering space theory outside of topology.

<u>Thm</u> If G is a free group of rank $n+1$ and H a subgroup of G of index k, then H is free of rank $nk+1$.

<u>Def</u> A <u>graph</u> is a finite CW complex of dimension at most 1.



Prop If Γ is a connected graph, then $\pi_1(\Gamma, v)$ is free of rank $1-\chi(\Gamma)$ for any vertex v.

Proof We proceed by induction on the number V of vertices. If $V=1$, then Γ is a wedge of $1-\chi(\Gamma)$ circles. For the induction step, since $V>1$, Γ has an edge homeomorphic to $[0,1]$. By the pasting lemma the straight line deformation retraction of $[0,1]$ onto $\{0\}$ extends to a homotopy equivalence between Γ and a graph with one fewer edge and the same Euler characteristic.

$\square$

Prop Let Γ be a graph and $E \xrightarrow{p} \Gamma$ a k-fold covering space. Then E is a graph, and $\chi(E) = k\chi(\Gamma)$.

Proof Without loss of generality, all edges of Γ are homeomorphic to $[0,1]$. Then for every edge $e \subseteq \Gamma$, $p^{-1}(e) \to e$ is a covering space of order k, hence a disjoint union of k intervals, since $\pi_1(e) = \{1\}$. Thus, E is a union of subspaces homeomorphic to $[0,1]$. The rest is an exercise. $\square$

Proof of theorem Let G be a free group of rank $n+1$, and choose a system of free generators to obtain an isomorphism $\varphi: \pi_1(\bigvee_{n+1} S^1, x_0) \cong G$. Given $H \leq G$, there is a path connected covering space $p: E \to \bigvee_{n+1} S^1$ such that $\varphi(p_*(\pi_1(E, e_0))) = H$ for some $e_0 \in E$. Since $[G:H] = k$, p is a k-fold cover, so E is a graph, and $\pi_1(E, e_0)$ is free of rank

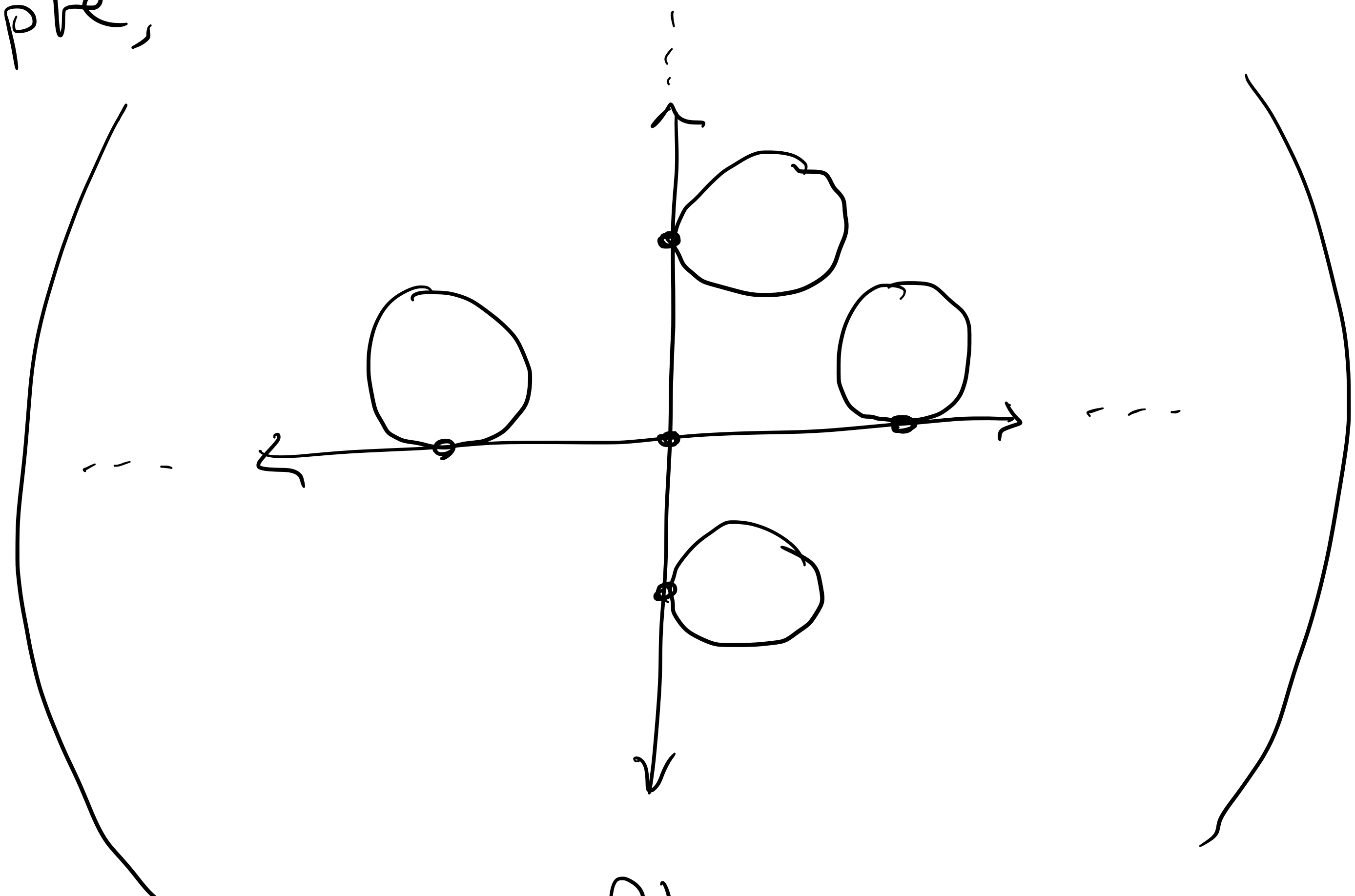
$$\begin{aligned} 1 - \chi(E) &= 1 - k\chi(\textstyle\bigvee_{n+1} S^1) \\ &= 1 - k(1 - (n+1)) \\ &= kn + 1. \end{aligned}$$

Since p_* and φ are injective, the claim follows.

$\square$

Rmk If $[G:H]$ is not finite, H is still free, but nothing can be said about its rank; it may be finite (e.g., $H = \{1\}$) or infinite.

For example,

$H = \pi_1$ 

$\cap$

$\pi_1(\infty)$

Returning to the problem of replacing maps by fibrations, we will be inspired by our study of covering spaces to construct a universal fibration over X. In fact, the construction will be very similar; we will again use a space of paths

$$X^{[0,1]} = \{\gamma : [0,1] \to X\}.$$

How to topologize?

Def The compact-open topology on $\mathrm{Map}(X,Y) =: Y^X := \{f : X \to Y \mid f \text{ continuous}\}$ is the topology generated by the subsets $\{f \mid f(K) \subseteq U\}$, where K ranges over the compact subsets of X and U over the open subsets of Y.

When dealing with mapping spaces, we will always assume that all spaces are Hausdorff, and that the source is also locally compact. This is not the maximal generality.

<u>Prop</u> (1) If Y is a metric space, then the compact-open topology is the topology of uniform convergence on compact subspaces.

(2) The following maps are continuous:

$$\begin{aligned} Y^X \times X &\xrightarrow{ev} Y \\ (f,x) &\longmapsto f(x) \end{aligned} \qquad \begin{aligned} Z^Y \times Y^X &\longrightarrow Z^X \\ (g,f) &\longmapsto g\circ f \end{aligned}$$

(3) The following natural bijections are homeomorphisms:

$$Z^{X\times Y} \cong (Z^Y)^X \qquad (Y\times Z)^X \cong Y^X \times Z^X \qquad Z^{X\amalg Y} \cong Z^X \times Z^Y$$

(4) Given $X \times T \xrightarrow{f} Y$, f is continuous iff $f(-,t)$ is continuous for each $t \in T$ as well as the function

$$\begin{aligned} T &\longrightarrow Y^X \\ t &\longmapsto f(-,t). \end{aligned}$$

<u>Proof</u> Bredon VII.2. $\square$

<u>Corollary</u> $\left\{ \begin{array}{l} \text{Homotopy classes} \\ \text{of maps } X \to Y \end{array} \right\} \cong \pi_0 Y^X.$

Equipping $X^{[0,1]}$ with the CO topology, the evaluation at $t=0$ map $X^{[0,1]} \xrightarrow{ev_0} X$ is continuous.

<u>Observation</u> ev_0 is a homotopy equivalence.

Indeed, the "constant path" map $X \to X^{[0,1]}$ is the embedding of a deformation retract via

$$X^{[0,1]} \times [0,1] \longrightarrow X^{[0,1]}$$
$$(\gamma, s) \longmapsto \gamma_s \qquad \gamma_s(t) = \gamma(st)$$

<u>Observation</u> ev_0 is a fibration.

Indeed, the following lifting problems are equivalent:

$$
\begin{tikzcd}
D^n \times \{0\} \arrow[r] \arrow[d] & X^{[0,1]} \arrow[d] \\
D^n \times [0,1] \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}
\iff
\begin{tikzcd}
D^n \times \{0\} \times [0,1] \cup D^n \times [0,1] \times \{0\} \arrow[d] \arrow[dr, bend left] & \\
D^n \times [0,1] \times [0,1] \arrow[r, dashed] & X
\end{tikzcd}
$$

and $\{0\} \times [0,1] \cup [0,1] \times \{0\} \subseteq [0,1] \times [0,1]$ is a retract.