Last time

- Construction of $p: E \to B$ w/ $p_*(\pi_1(E, e_0)) = H$
- Subgroups of free groups
- Compact-open topology

We ended by showing that, in the diagram

$$\begin{tikzcd} X \arrow[r, "\text{const}"] \arrow[dr, "id"'] & X^{[0,1]} \arrow[d, "ev_1"] \\ & X \end{tikzcd}$$

the top map is the inclusion of a deformation retract and the right hand map a fibration.

<u>Def</u> The mapping path space of $f: X \to Y$ is

$$P_f := X \times_Y Y^{[0,1]} = \{(x,\gamma) \in X \times Y^{[0,1]} \mid \gamma(0) = f(x)\}.$$

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{[0,1]} \arrow[d, "ev_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

<u>Prop</u> In the diagram

$$x \longmapsto (x, \underline{f(x)})$$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & P_f \arrow[dl] \\ & Y & \end{tikzcd} \qquad (x,\gamma) \longmapsto \gamma(1)$$

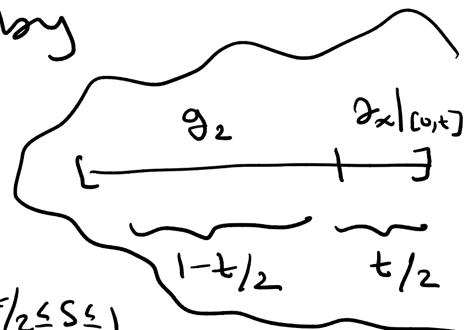
all maps are continuous, the top map is a homotopy equivalence, and the right hand map a is fibration.

Proof We prove only the last claim. Consider the lifting problem

$$\begin{tikzcd}
D^n \times \{0\} \arrow[r, "{(g_1,g_2)}"] \arrow[d] & X \times_Y Y^{[0,1]} \arrow[d, "ev_1"] \\
D^n \times [0,1] \arrow[r, "H"] \arrow[ur, dashed, "\exists \tilde{H}?"] & Y
\end{tikzcd}$$

For $x \in D^n$, write $\gamma_x = H(x,-) : [0,1] \to Y$ and define $\tilde{g}_2 : D^n \times [0,1] \longrightarrow Y^{[0,1]}$ by

$$\tilde{g}_2(x,t)(s) = \begin{cases} g_2(x)\left(\frac{s}{1-t/2}\right) & 0 \le s \le 1-t/2 \\ \gamma_x\left(2\left(s-(1-t/2)\right)\right) & 1-t/2 \le s \le 1 \end{cases}$$



Note that $\gamma_x(0) = H(x,0) = g_2(x)(1)$, so this function is well-defined, and continuity is an exercise in the CO topology. We now define $\tilde{H}(x,t) = (g_1(x), \tilde{g}_2(x,t))$. Since $\tilde{g}_2(x,t)(0) = g_2(x)(0) = g_1(x)$, $\tilde{H}$ defines a map into P_f. Finally, we check commutativity:

$$\tilde{H}(x,0) = (g_1(x), \tilde{g}_2(x,0)) = (g_1(x), g_2(x))$$

$$\tilde{g}_2(x,t)(1) = \gamma_x(t) = H(x,t).$$

$\square$

Def The homotopy fiber of $f: X \to Y$ (over $y \in Y$), denoted $\mathrm{hofib}(f)$, is the fiber of $P_f \xrightarrow{ev_1} Y$.

Cor For any $f: X \to Y$, there is an exact sequence

$$\cdots \to \pi_n(\mathrm{hofib}(f)) \to \pi_n(X) \to \pi_n(Y) \to \cdots$$

Ex If $f: pt \to X$ is the inclusion of x_0, then $P_f = (X, x_0)^{([0,1], \{0\})} =: PX$, the path space of X (at x_0), and the homotopy fiber (at x_0) is $(X, x_0)^{([0,1], \{0,1\})} =: \Omega X$, the <u>loop space</u> of X (at x_0).

<u>Cor</u> $\pi_1(X) \cong \pi_0(\Omega X)$. More generally, $\pi_n(X) \cong \pi_{n-1}(\Omega X)$. In particular, $\pi_n(X) \cong \pi_0(\Omega^n X)$, where

$$\Omega^n X = \Omega \Omega^{n-1} X \cong (X, x_0)^{([0,1]^n, \partial)} \cong (X, x_0)^{(S^n, *)}.$$

<u>Cor</u> $\Omega K(G, n+1) \simeq K(G, n)$ for $n > 0$.

<u>Cor</u> $K(G, n) \simeq \Omega^r K(G, n+r)$ for $n > 0$, $r \geq 0$.

<u>Upshot</u> $\pi_4(S^3) \cong H_4(X)$, where $X = \mathrm{hofib}(S^3 \to K(\mathbb{Z},3))$.

<u>Plan</u> Calculate $H_*(K(\mathbb{Z},3))$ using the fiber sequence

$$\mathbb{C}P^\infty \simeq K(\mathbb{Z},2) \to pt \longrightarrow K(\mathbb{Z},3),$$

then calculate $H_4(X)$ using the fiber sequence

$$X \longrightarrow S^3 \longrightarrow K(\mathbb{Z},3).$$

<u>Problem</u> What do fibrations have to do with (co)homology?

For example, our "plan" suggests that we should be able to express $H_*(\mathbb{CP}^\infty)$ in terms of $H_*(S^1)$ via the Hopf fibration

$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{CP}^\infty.$$

That sounds hard!

<u>Strategy</u> Given a CW complex B and a fibration $\pi : E \to B$, try to understand the total space as the union $E = \bigcup_{p \geq 0} E_p$, where $E_p = \pi^{-1}(B_p)$ (the p-skeleton of B).

Def A *filtration* of a space X is a collection of subspaces $\{X_p\}_{p \in \mathbb{Z}}$ with $X_p \subseteq X_{p+1}$ and $X = \bigcup_{p \in \mathbb{Z}} X_p$.

Example The skeleta of a CW complex form a filtration.

This example and our experience with cellular (co)homology motivates the following.

Hope A filtration is useful in calculating (co)homology.

Assume for simplicity that (X_p, X_{p-1}) is a good pair for every p, and that $X_{-1} = \phi$. A first approximation to $H_*(X)$ in terms of the filtration is the direct sum

$$\bigoplus_p H_*(X_p) \longrightarrow H_*(X)$$

Since every $\sigma : \Delta^n \to X$ factors through some X_p, this map is surjective, but there is a lot of redundancy. A better approximation is the direct sum

$$E' := \bigoplus_p \tilde{H}_*(X_p/X_{p-1}).$$

<u>Example</u> This approximation is correct for <u>split</u> filtrations $X = \bigvee_n Y_n$,

$$X_p = \bigvee_{n \le p} X_p.$$

To understand how far E' is from $\tilde{H}_*(X)$, consider $[c] \in H_*(X_p, X_{p-1})$:

$$\begin{array}{ccccc} H_*(X_p, X_{p-1}) & \longleftarrow & H_*(X_p) & \longrightarrow & H_*(X) \\ [c] & \overset{?}{\longleftarrow\!\shortmid} & [\tilde{c}] & \overset{?}{\longmapsto} & \neq 0 \end{array}$$

For the first question note that ∂c consists of simplices with image in X_{p-1}, and $\tilde{c}$ exists iff c may be chosen so that the image lies in X_{-1}, i.e., so that $\partial c = 0$. A necessary condition, then, is that c may be chosen so that ∂c lies in X_{p-2}, i.e., $[c] \in \ker(d')$, where d' is the map

$$\begin{tikzcd}
\tilde{H}_n(X_p/X_{p-1}) \arrow[r, "\delta"] \arrow[dr, "d_p'"'] & H_{n-1}(X_{p-1}) \arrow[d, "q_*"] \\
& \tilde{H}_{n-1}(X_{p-1}/X_{p-2})
\end{tikzcd}$$

For the second question, for $[c]$ to be nonzero in $H_*(X)$, a necessary condition is that $[c]$ is nonzero in $H_*(X_{p+1})$, i.e., $[c]$ is not the boundary of a chain in X_{p+1}, i.e., $[c]$ is not in the image of

$$\begin{tikzcd}
\tilde{H}_n(X_{p+1}/X_p) \arrow[r, "\delta"] \arrow[dr, "d^1_{p+1}"'] & H_{n-1}(X_p) \arrow[d, "q_*"] \\
& \tilde{H}_{n-1}(X_p/X_{p-1})
\end{tikzcd}$$

Thus, a better approximation than

E^1 is

$$E^2 := \bigoplus_p \ker(d^1_p) / \operatorname{im}(d^1_{p+1}),$$

i.e., the homology of the chain complex (E^1, d^1).

<u>Example</u> Filtering a CW complex by skeleta, $E^1 = C^{CW}_*(X)$, $d^1 = d^{CW}$, and $E^2 = H^{CW}_*(X) \cong H_*(X)$.

This example shows that E^2 is sometimes an accurate approximation, but this need not happen in general. In this case, one continues to ask the same questions: can we choose c such that ∂c lies in X_{p-3} and so that $[c]$ is nonzero in X_{p+2}? A differential d^2 again captures the answers, leading to E^3, and so on. The sequence $\{(E^r, d^r)\}$ is called a spectral sequence.