Last time

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[dr, "f"'] & P_f \arrow[d] \\ & Y \end{tikzcd}$$

- Mapping path space
- (co)homology of fibrations?
- Filtrations $\cdots \subseteq X_p \subseteq X_{p+1} \subseteq \cdots$, $X = \bigcup_p X_p$.
- Approximations to $H_*(X)$.

Recall our successive approximating

$$\bigoplus_p H_*(X_p)$$

$$\bigoplus_p H_*(X_p, X_{p-1}) =: E^1$$

$$\bigoplus_p \ker d^1_p / \operatorname{im} d^1_{p+1} =: E^2$$

$$\begin{tikzcd} H_n(X_p, X_{p-1}) \arrow[r, "\delta"] \arrow[dr, "d^1_p"'] & H_{n-1}(X_{p-1}) \arrow[d] \\ & H_{n-1}(X_{p-1}, X_{p-2}) \end{tikzcd}$$

The relationships among these are summarized pictorially as follows:

$$\begin{tikzcd}
\bigoplus_p H_*(X_p) \arrow[rr, "\bigoplus (i_p)_*"] & & \bigoplus_p H_*(X_{p+1}) \arrow[dl, "\bigoplus (q_p)_*"] \\
& \bigoplus_p H_*(X_{p+1}, X_p) \arrow[ul, "\bigoplus_p \delta"] &
\end{tikzcd}$$

This diagram does <u>not</u> commute. Rather, it is exact at each corner.

<u>Def</u> An <u>exact couple</u> is a diagram

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

of Abelian groups such that $\ker j = \operatorname{im} i$, $\ker k = \operatorname{im} j$ and $\ker i = \operatorname{im} k$.

The main source of exact couples is filtered chain complexes.

<u>Def</u> A <u>filtration</u> of a chain complex C is a collection of subcomplexes $C_p \subseteq C$, $p \in \mathbb{Z}$, with $C_p \subseteq C_{p+1}$ and $C = \bigcup_p C_p$.

__Ex__ If X_p is a filtration of X, then $C_*(X)_p := C_*(X_p)$ is a filtration of $C_*(X)$.

__Ex__ If $C = \bigcup_p C_p$ is a filtered chain complex, then

$$\begin{tikzcd}
\bigoplus_p H_*(C_p) \arrow[rr] & & \bigoplus_p H_*(C_{p+1}) \arrow[dl] \\
& \bigoplus_p H_*(C_{p+1}/C_p) \arrow[ul] &
\end{tikzcd}$$

is an exact couple.

In our original example, the differential d' is the composite of the two diagonal maps.

<u>Observation</u> Given an exact couple as above, and setting $d = jk$, we have

$$d^2 = jkjk = 0,$$

since $kj = 0$ by exactness. Thus, (E, d) is a chain complex.

Def/Prop The *derived couple* is the exact couple

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd},$$

where

(1) $E' = \ker d / \operatorname{im} d$

(2) $A' = \operatorname{im} i \subseteq A$

(3) $i' = i|_{A'}$

(4) $j'(i(a)) = [j(a)] \in E'$

(5) $k'([e]) = k(e)$.

Proof First, j' is well-defined, since

$$\begin{aligned}\bar{i}(a)=\bar{i}(b) &\iff a-b \in k(C)\\ &\implies j(a)-j(b) = jk(c) = dc\\ &\implies [j(a)] = [j(b)].\end{aligned}$$

Second, k' is well-defined. There are two points to check:

$$\begin{tikzcd} \ker(d) \subseteq E \arrow[r, "k"] \arrow[d] \arrow[ddr, dashed, "1"] & A \\ & \rotatebox{90}{$\supseteq$} \\ E' \arrow[r, dashed, "2"] & A' \end{tikzcd}$$

For the first,

$$de = 0 \iff jk(e) = 0$$
$$\iff k(e) \in \ker j = \operatorname{im} i = A'.$$

For the second, $k(d\tilde{e}) = kjk(\tilde{e}) = 0$, since $kj = 0$.

Next, we check exactness at E, leaving the rest as an exercise. We have

$$\begin{aligned} k'j'i(a) &= k'([j(a)]) \\ &= kj(a) \\ &= 0, \end{aligned}$$

so $\operatorname{im} j' \subseteq \ker k'$. Conversely,

$$k'([e]) = 0 \iff k(e) = 0$$
$$\iff e \in \operatorname{im} j$$
$$\implies [e] = [j(a)] = j'(i(a)),$$

so $\ker k' \subseteq \operatorname{im} j'$. $\square$

We now iterate this process. Setting $E = E^1$ and $E^r = (E^{r-1})'$, we obtain chain complexes (E^r, d^r), each being the homology of the last.

<u>Warning</u> (E^r, d^r) determines E^{r+1} but not d^{r+1}!

Def A spectral sequence is a sequence $\{(E^r, d^r)\}$ of homomorphisms

$$d^r : E^r \longrightarrow E^r$$

such that $(d^r)^2 = 0$ and $E^{r+1} = H_*(E^r, d^r)$.

To summarize the discussion so far:

filtered space $\rightsquigarrow$ filtered complex $\rightsquigarrow$ exact couple $\rightsquigarrow$ spectral sequence

Hope As $r \to \infty$, E^r "converges" to $H_*(X)$.

To understand in what sense convergence can occur, it will be helpful to track gradings. In the situation of a filtered chain complex, the components of the exact couple are bigraded:

$$A^1 = \bigoplus_{p,q} A^1_{p,q} \qquad E^1 = \bigoplus_{p,q} E^1_{p,q}$$

$$A^1_{p,q} = H_{p+q}(C_p) \qquad E^1_{p,q} = H_{p+q}(C_p / C_{p-1})$$

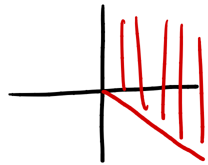
The maps in the exact couple respect the bigrading in that

$$\left.\begin{aligned} i(A^1_{p,q}) &\subseteq A^1_{p+1,q-1} \\ j(A^1_{p,q}) &\subseteq E^1_{p,q} \\ k(E^1_{p,q}) &\subseteq A^1_{p-1,q} \end{aligned}\right\} \quad d^1_{p,q}(E^1_{p,q}) \subseteq E^1_{p-1,q}$$

Pictorially:

$$\begin{array}{c|ccccc} q & & & \vdots & & \\ & E^1_{0,2} & \leftarrow E^1_{1,2} & \leftarrow & E^1_{2,2} & \\ & E^1_{0,1} & \leftarrow E^1_{1,1} & \leftarrow & E^1_{2,1} & \cdots \\ & E^1_{0,0} & \leftarrow E^1_{1,0} & \leftarrow & E^1_{2,0} & \\ \hline & & & & & p \end{array}$$

Assuming that $C_{-1} = 0$ and that C is non-negatively graded, $A^1_{p,q} = E^1_{p,q} = 0$ for $p<0$ or $q<-p<0$ so the spectral sequence is concentrated in the region $\{p>0\} \cap \{p+q>0\}$



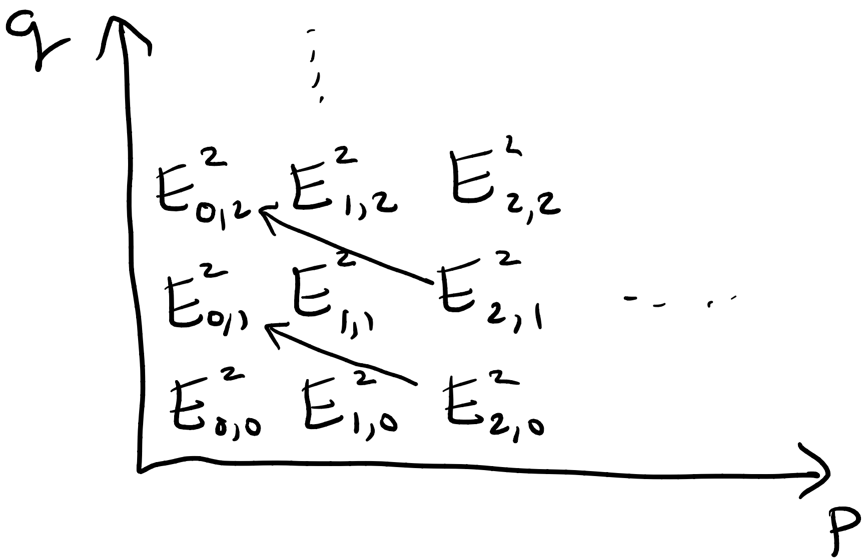
Since i, j, and k respect gradings, we obtain decompositions

$$A^2 = \bigoplus A^2_{p,q} \qquad E^2 = \bigoplus E^2_{p,q}$$

$$A^2_{p,q} = A^2 \cap A^1_{p,q} \qquad E^2_{p,q} = \frac{\ker d^1_{p,q}}{\operatorname{im} d^1_{p+1,q}}.$$

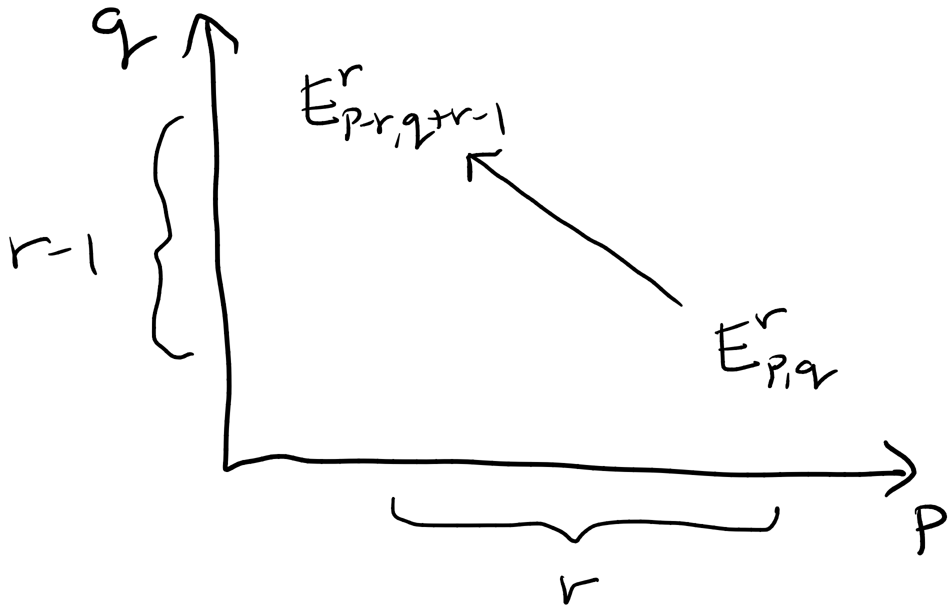
Now however, we have

$$\left.\begin{aligned} i'(A^2_{p,q}) &\subseteq A^2_{p+1,q-1} \\ j'(A^2_{p,q}) &\subseteq E^2_{p-1,q+1} \\ k'(E^2_{p,q}) &\subseteq A^2_{p-1,q} \end{aligned}\right\} \quad d^2_{p,q}(E^2_{p,q}) \subseteq E^2_{p-2,q+1}$$



Similarly, we have that

$$d^r_{p,q}(E^r_{p,q}) \subseteq E^r_{p-r,q+r-1}$$



Def A filtration $C = \bigcup_p C_p$ of a chain complex is *eventually locally finite* if, for each $n \geq 0$, there exists $r(n)$ such that $E^{r(n)}_{p,n-p} = 0$ for all but finitely many p. If $r(n) \equiv 1$, then it is *locally finite*. It is *complete* if $C_p \to C$ induces an isomorphism on H_n for $p \gg 0$ and *bounded below* if $C_p = 0$ for some p.

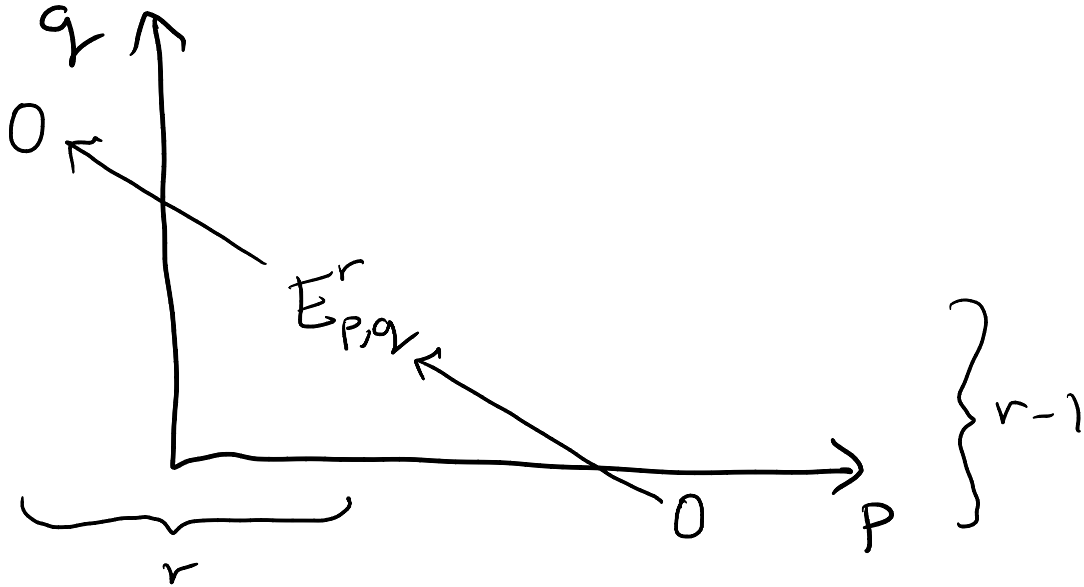
Exercise Complete + bounded below $\Rightarrow$ LF

Ex A filtration is LF if it gives rise to a *first-quadrant* spectral sequence, i.e., if $E^r_{p,q} = 0$ for $q < 0$.

<u>Lemma</u> If $C = \bigcup_p C_p$ is ELF, then $d^r_{p,q} \equiv 0 \equiv d^r_{p+r,q-r+1}$ for fixed p, q and r sufficiently large.

<u>Proof</u> Fix p_0 and q_0 and set $p_0 + q_0 = n$. The target of $d^r_{p_0,q_0}$ lies on the line $p+q = n-1$, and the source of $d^r_{p_0+r,q_0-r+1}$ on the line $p+q = n+1$, for every r. By assumption, these lines contain finitely many nonzero entries, so

the respective target and source vanish for r sufficiently large. $\square$



<u>Cor</u> If $C = \bigcup_p C_p$ is ELF, then $E^r_{p,q} = E^{r+1}_{p,q} = \cdots$ for fixed p,q and r sufficiently large.

Write $E^{\infty}_{p,q} := E^r_{p,q}$ for $r \gg 0$.

<u>Lemma</u> The filtration $C = \bigcup_p C_p$ is ELF (resp. LF) iff $A^r_{p,n-p} \xrightarrow{\cong} A^r_{p+1,n-p-1}$ for all but finitely many p and r sufficiently large relative to n (resp. A').

<u>Proof</u> The claim follows from the exact sequence

$$E^r_{p+1,n-p} \to A^r_{p,n-p} \to A^r_{p+1,n-p-1} \to E^r_{p-r+2,n-p+r-2}.$$

$\square$

Write $i^{\infty}(A'_{p,q}) := i^{r-1}(A'_{p,q}) = A^r_{p+r-1,q-r+1}$ for $r \gg 0$.

<u>Thm</u> (Convergence) Let $C = \bigcup_p C_p$ be ELF and bounded below. Then

$$E^\infty_{p,q} \cong i^\infty(A^1_{p,q}) \big/ i^\infty(A^1_{p-1,q+1}).$$

If C is also complete, then

$$i^\infty(A^1_{p,q}) \cong \operatorname{im}\left(H_{p+q}(C_p) \longrightarrow H_{p+q}(C)\right).$$