Last time

- Hopf bundles: $\pi_*(\mathbb{C}P^\infty)$, $\pi_3(S^2) \neq 0$
- Connectivity, Hurewicz
- $\pi_n(S^n) \cong \mathbb{Z}$ via the degree

Lemma If X is n-connected, then the inclusion $C_*^{(n)}(X) \subseteq C_*(X)$ is a chain homotopy equivalence.

Thm (Hurewicz) If X is $(n-1)$-connected for some $n \geq 1$, then

$$h_k : \pi_k(X) \longrightarrow H_k(X)$$

is an isomorphism for every $0 < k \leq n$.

Proof of thm. The lemma directly implies that $H_k(X)=0$ for $k<n$, since (exercise) $H_k^{(n-1)}(X)=0$ in this range, so it suffices to show that h_n is an isomorphism. Under the identification $S^n \cong \Delta^n/\partial\Delta^n$, a pointed map sends $\partial\Delta^n$ to x_0, so defines an element of $H_n^{(n-1)}(X)$. Thus we have the factorization below.

$$\begin{tikzcd}
\pi_n(X) \arrow[rr, "h_n"] \arrow[dr, dashed, "\bar{h}_n"'] & & H_n(X) \\
& H_n^{(n-1)}(X) \arrow[ur, "\cong\ \text{lem.}"'] &
\end{tikzcd}$$

Conversely, there is a map

$$\varphi : \tilde{C}_n^{(n-1)}(X) \longrightarrow \pi_n(X)$$
$$f:(\Delta^n,\partial\Delta^n)\to(X,x_0) \longmapsto [f].$$

and, since $\partial|_{\tilde{C}_n^{(n-1)}(X)} \equiv 0$, we obtain a homomorphism $\overline{\varphi}: H_n^{(n-1)}(X) \to \pi_n(X)$ upon checking that φ annihilates boundaries.

Given an $(n+1)$-simplex $\tau: \Delta^{n+1} \to X$ taking the $(n-1)$-skeleton to x_0, it is clear that $[\tau|_{\partial\Delta^{n+1}}] = 0 \in \pi_n(X)$; therefore, it suffices to observe that

$\varphi(\partial\tau) = [\tau|_{\partial\Delta^{n+1}}]$ by a variant of the argument showing that higher homotopy groups are commutative.

$\square$

What about $\pi_k(S^n)$ for $k>n$?

<u>Ex</u> In the case of homology, we have the suspension isomorphism $H_k(S^n) \cong H_{k+1}(S^{n+1})$.

For homotopy groups, we have the analogous homomorphism

$$\pi_k(X) \xrightarrow{S_k} \pi_{k+1}(SX)$$
$$[f: S^k \to X] \longmapsto [Sf: S^{k+1} \to SX].$$

<u>Thm</u> (Freudenthal) If X is an $(n-1)$-connected CW complex, then S_k is an isomorphism for $k < 2n-1$ and S_{2n-1} is a surjection.

We defer the proof to later in the course.

Ex It follows that $\mathbb{Z} \cong \pi_3(S^2)$ surjects onto $\pi_4(S^3) \cong \pi_5(S^4) \cong \cdots$. As we will see $\pi_{n+1}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 2$.

Cor $\pi_{n+r}(S^n)$ is independent of n for $n \geq r+2$

Proof The theorem implies that S_{n+r} is an isomorphism for $n+r < 2n-1 \Leftrightarrow n \geq r+2$.

$\square$

This corollary leads to the study of the stable homotopy groups of spheres.

Q How powerful are homotopy groups?

Ex For $X = \mathbb{RP}^2, S^2 \times \mathbb{RP}^\infty$

$$\pi_i(X) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & i=1 \\ \pi_i(S^2), & i>1, \end{cases}$$

but homology shows the two are not homotopy equivalent.

If we require the existence of a map, the situation changes.

Def A map $f: X \to Y$ is a <u>weak equivalence</u> if $f_*: \pi_i(X) \to \pi_i(Y)$ is an isomorphism for every $i \geq 0$.

__Ex__ A homotopy equivalence is a weak equivalence.

__Ex__ The identity map $\mathbb{R}_s \xrightarrow{id} \mathbb{R}_e$ is a weak equivalence, where $\mathbb{R}_s$ carries the discrete topology and $\mathbb{R}_e$ the lower limit topology; indeed, both spaces are totally disconnected, so π_0 is simply the set of points and $\pi_i = 0$ for $i > 0$. But there is no continuous bijection $\mathbb{R}_e \to \mathbb{R}_s$, hence no map inducing a bijection on π_0.

This example is a pathology, as the next result shows.

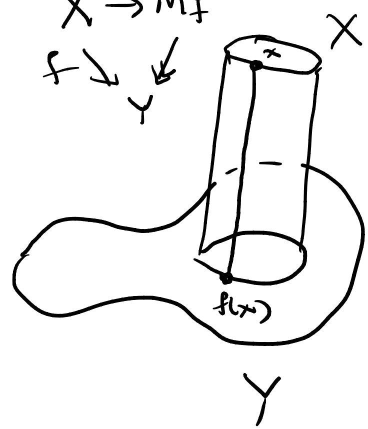
<u>Thm</u> (Whitehead) Let $f: X \to Y$ be a weak equivalence. If X and Y are CW complexes, then f is a homotopy equivalence. Moreover if f is the inclusion of a subcomplex, then this subcomplex is a deformation retract.

<u>Cor</u> If a map between simply connected CW complexes induces isomorphisms on homology in every degree, then it is a homotopy equivalence.

Def The <u>mapping cylinder</u> of $f: X \to Y$ is the space

$$M_f = \frac{X \times [0,1] \amalg Y}{(x,1) \sim f(x)}$$

$$\begin{array}{ccc} X & \overset{\subseteq}{\to} & M_f \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & Y & \end{array}$$



Exercise The inclusion $Y \subseteq M_f$ is a deformation retract.

Exercise The pair (M_f, X) is a good pair, so homotopies defined on X extend to M_f.

Lemma f is a weak equivalence iff $\pi_n(M_f, X) = 0$ for every $n \geq 0$.

<u>Proof</u>

$$
\begin{tikzcd}
\cancelto{0}{\pi_{n+1}(M_f, X)} \arrow[r] & \pi_n(X) \arrow[rr] \arrow[dr, "f_*"'] & \arrow[d, phantom, "\circlearrowleft"] & \pi_n(M_f) \arrow[r] \arrow[dl, "\cong"] & \cancelto{0}{\pi_{n-1}(M_f, X)} \\
& & \pi_n(Y) & &
\end{tikzcd}
$$

$\square$