Last time

- Proof of Hurewicz
- Freudenthal and stable homotopy groups
- Weak equivalences
- Whitehead
- Mapping cylinders

Thm (Whitehead) A weak equivalence between CW complexes is a homotopy equivalence. A weakly equivalent subcomplex is a deformation retract.

Lemma (compression) Let (X,A) be a CW pair and (Y,B) any pair with $B \neq \emptyset$. Suppose that $\pi_n(Y,B,y_0)=0$ for every $y_0 \in B$ and n such that X has an n-cell not lying in A. Every map $f:(X,A)\to(Y,B)$ is homotopic rel A to a map with image lying in B.

Proof Assume inductively that $f(X_{k-1}) \subseteq B$. For each cell not in A, the composite

$$(D^k, \partial D^k) \to (X, X_{k-1}) \to (Y, B)$$

is homotopic rel ∂D^k to a map with image in B (we use our assumption. Extending by the constant homotopy on A, we obtain a homotopy defined on $X_k \cup A$, which is a subcomplex of X, so the homotopy extends. Performing this homotopy in the interval $[1-1/2^k, 1-1/2^{k+1}]$ gives the desired homotopy. $\square$

Proof of thm The special case of a subcomplex is immediate from the compression lemma. In the general case, assume that $f: X \to Y$ is a weak equivalence, so $\pi_n(M_f, X) = 0$. We will show that $X \subseteq M_f$ is a deformation retract. It then follows that f is the composite of two homotopy equivalences, implying the claim.

First, apply the lemma to the inclusion $(X \cup Y, X) \hookrightarrow (M_f, X)$ to see that id_{M_f} is homotopic rel X to a map g taking Y into X (we use that $(M_f, X \cup Y)$ is good).

Next, apply the lemma to the map

$$(X \times [0,1] \amalg Y,\ X \times \{0,1\} \amalg Y) \to (M_f, X \cup Y) \xrightarrow{g} (M_f, X)$$

to obtain a homotopy rel $X \times \{0\}$ that descends to a homotopy rel X from g to a map with image lying in X.

$\square$

Cor If a map between simply connected CW complexes induces isomorphisms on homology in every degree, then it is a homotopy equivalence.

Proof The assumption on π_1 and a relative version of the Hurewicz theorem show that $\pi_n(M_f, X) = 0$ for every n, so f is a weak equivalence, hence a homotopy equivalence by Whitehead. $\square$

Rk Not every space is homotopy equivalent to a CW complex; indeed, our earlier example shows that $\mathbb{R}_\ell$ is not (since $\mathbb{R}_\delta$ is).

Rk (CW approximation) Every space is weakly equivalent to a CW complex.

Rk (Cellular approximation) Every map between CW complexes is homotopic to a cellular map.

Rk If f is cellular then (M_f, X) is a CW pair. Thus, up to weak equivalence, every map may be replaced by the inclusion of a subcomplex.

Now that we know homotopy groups are "worth it," let's return to the problem of calculating them, say $\pi_4(S^3)$ for concreteness.

<u>Idea</u> We can calculate homology because we have cellular chains, which reduce the calculation to considerations of spaces with <u>only one nonzero homology group</u>.

__Def__ Fix $n > 0$ and a group G, Abelian if $n \neq 1$. An __Eilenberg-MacLane space__ of type (G,n) is a space $K(G,n)$ with

$$\pi_k K(G,n) \cong \begin{cases} G & k = n \\ 0 & \text{otherwise.} \end{cases}$$

__Ex__ $S^1 = K(\mathbb{Z},1)$, $\mathbb{CP}^\infty = K(\mathbb{Z},2)$, $\mathbb{RP}^\infty = K(\mathbb{Z}/2\mathbb{Z},1)$.

__Thm__ For any (G,n), an Eilenberg-MacLane space of type (G,n) exists, and any two are weakly equivalent.

Assuming this for now, we have a map

$$S^3 \longrightarrow K(\mathbb{Z},3)$$

representing a generator of $\pi_3(K(\mathbb{Z},3)) = \mathbb{Z}$, which induces an isomorphism on π_3.

Let's imagine this map were a fibration; then the fiber X has

$$\pi_n(X) \cong \begin{cases} \pi_n(S^3) & n>3 \\ 0 & n \leq 3, \end{cases}$$

so $\pi_4(S^3) \cong H_4(X)$ by Hurewicz. Thus, we may reduce homotopy to homology given the following ingredients:

(1) Eilenberg-MacLane spaces

(2) replacements of maps by fibrations

(3) relations among the homology groups of base, total space, and fiber in a fibration.

We take up (1) presently, (2) will be lead us to path and loop spaces, and (3) is the Serre spectral sequence.

To get started, we require the following fundamental result.

Prop If X is a CW complex, then $\pi_k(X, X_n) = 0$ for $k \leq n$.

Lemma 0 If X is a CW complex and $X_n \subseteq A \subseteq X$, then $\pi_k(X, A) = 0$ for $k \leq n$.

Proof From last time, $\pi_k(X, X_n) = 0$, so the top map in the diagram

$$\begin{tikzcd} \pi_k(X_n) \arrow[rr] \arrow[dr] & & \pi_k(X) \\ & \pi_k(A) \arrow[ur, "i_*"'] & \end{tikzcd}$$

is surjective, whence i_* is so. $\square$