Last time

- Proof of Whitehead
- Strategy for $\pi_4(S^3)$
- Eilenberg-MacLane spaces

$$X \to S^3 \to K(\mathbb{Z},3)$$

$$\pi_4(S^3) \cong H_4(X)$$

Prop If X is a CW complex, then $\pi_k(X, X_n) = 0$ for $k \leq n$.

For this result, we require a little analysis.

Thm (Whitney approximation) Let $U \subseteq \mathbb{R}^k$ be an open subset and $A \subseteq U$ a closed subset. Given a map $f: U \to \mathbb{R}^m$, there is a homotopy $H: U \times [0,1] \to \mathbb{R}^m$ such that $H(-,0) = f$, $H(-,1)$ is smooth on $U \setminus A$, and $H(a,-)$ is constant for $a \in A$.

Proof Bredon II.11.7. $\square$

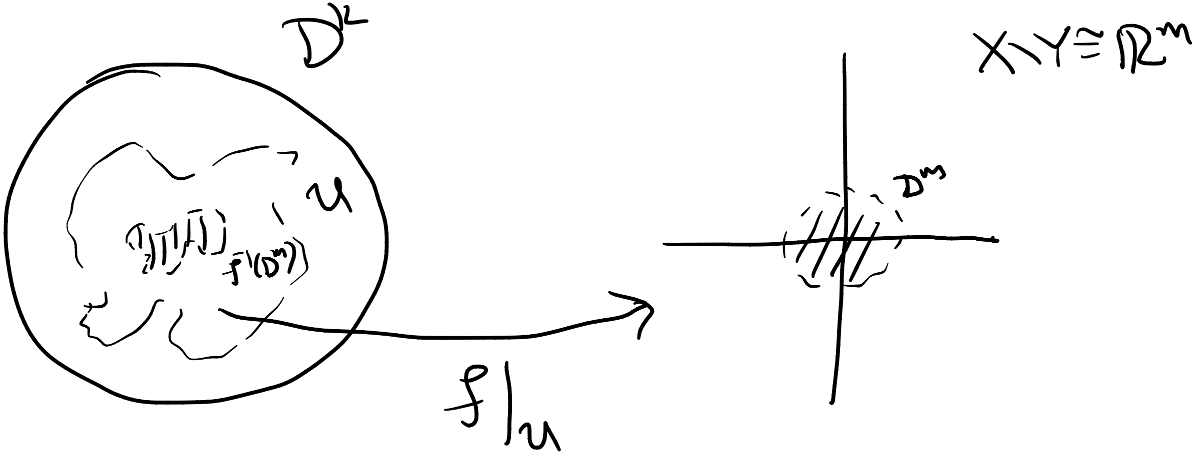
Thm (Sard) A non-constant smooth map $f: \mathbb{R}^k \supseteq U \to \mathbb{R}^m$ has a regular value.

Cor If $k < m$, then f is not surjective.

<u>Lemma</u> If X is obtained from Y by attaching an m-cell, then any map of pairs $f: (D^k, S^{k-1}) \to (X, Y)$ with $k < m$ factors through Y up to homotopy rel S^{k-1}.

<u>Proof</u> Since $X \setminus Y \cong \mathbb{R}^m$, we may apply Whitney to $f|_U$, where $U = f^{-1}(X \setminus Y)$, with $A = U \setminus f^{-1}(\mathring{D}^m)$. By the gluing lemma, since the resulting homotopy is constant on A, we may assume that $f|_{U \setminus A}$ is smooth,

hence $f|_{f^{-1}(\mathring{D}^m)}$ is not surjective. WLOG the origin in D^m does not lie in the image of f



So f factors through $Y \cup D^m \setminus \{0\}$, which deformation retracts onto Y. $\square$

<u>Proof of proposition</u> Let $f:(D^k, S^k) \to (X, X_n)$ be a map with $k \leq n$. Since D^k is compact, we may assume that X is obtained from X_n by attaching finitely many cells of dimension $> n \geq k$. Apply the lemma to each of these cells, and use the compression criterion. $\square$

<u>Exercise</u> Use the proposition to prove cellular approximation.

We return to the existence and uniqueness of $K(G,n)$ spaces.

Thm For any (G,n), an Eilenberg-MacLane space of type (G,n) exists, and any two are weakly equivalent.

Lemma 0 If X is a CW complex and $X_n \subseteq A \subseteq X$, then $\pi_k(X,A)=0$ for $k \leq n$.

Observation $\pi_n(\bigvee_I S^n) \cong \bigoplus_I \mathbb{Z}$ via the degree by Hurewicz.

<u>Lemma 1</u> For any (G, n), there is an $(n-1)$-connected CW complex X with $\pi_n(X) \cong G$.

<u>Proof</u> The case $n=1$ is a well-known consequence of the Van Kampen theorem, so assume $n>1$. There is a (non-unique) exact sequence

$$\bigoplus_J \mathbb{Z} \xrightarrow{\psi} \bigoplus_I \mathbb{Z} \xrightarrow{\varphi} G \longrightarrow 0$$

where I is a set of generators for G and J a set of generators for $\ker(\varphi)$. Write

$A = \bigvee_I S^n$ and $X = A \amalg_f \coprod_J D^{n+1}$, where $f : \coprod_J S^n \longrightarrow A$ is given on the j^{th} component by a representative for $\psi(1_j) \in \bigoplus_I \mathbb{Z} \cong \pi_n(A)$. Hurewicz gives the isomorphisms in the commuting diagram

$$\begin{tikzcd}
\pi_{n+1}(X,A) \arrow[r] \arrow[d, "\cong"] & \pi_n(A) \arrow[r] \arrow[d, "\cong"] & \pi_n(X) \arrow[r] \arrow[d] & \cancel{\pi_n(X,A)}^{\,0} \arrow[d] \\
H_{n+1}(X,A) \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & \cancel{H_n(X,A)}^{\,0}
\end{tikzcd}$$

By the five lemma, $\pi_n(X) \cong H_n(X)$, but $H_n(X) \cong G$ by cellular homology. $\square$

Lemma 2 (attaching a cell) Fix a CW complex X and $f: S^n \to X$. Setting $Y = X \cup_f D^{n+1}$, the map $\pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k<n$ and a surjection for $k=n$ with kernel containing $[f]$.

Proof Since $Y_n = X_n \subseteq X \subseteq Y$, $\pi_k(Y,X)=0$ for $k\leq n$ by Lemma 0. Since the composite $S^n \xrightarrow{f} X \subseteq Y$ extends over D^{n+1} by construction, it is null, implying the claim. $\square$

Lemma 3 (Killing homotopy) Given a CW complex X, there is a CW complex Y obtained from X by attaching cells of dimension $> n$ such that $\pi_k(Y) = 0$ for $k \geq n$ and the map $\pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$.

Proof Attaching $(n+1)$-cells along a generating set for $\pi_n(X)$ to obtain a space $X \subseteq Y^{(1)}$, Lemma 2 implies that $\pi_k(X) \to \pi_k(Y^{(1)})$ is an isomorphism for $k<n$ and a surjection for $k=n$ whose kernel is all of $\pi_n(X)$, so $\pi_n(Y^{(1)})=0$. Continuing in this way, we obtain a sequence of spaces

$$X \subseteq Y^{(1)} \subseteq Y^{(2)} \subseteq \cdots$$

and we set $Y = \bigcup_{r\geq 0} Y^{(r)}$. Since each skeleton of Y is contained in some $Y^{(r)}$, Lemma 0 implies that Y has the desired properties. $\square$

Lemma 4 (extension) Let (X, A) be a CW pair and $f: A \to Y$ a map. If $\pi_{n-1}(Y) = 0$ for every n such that X has a cell of dimension n not in A, then the following extension exists:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Proof Exercise. $\square$

Proof of thm By Lemma 1, there is an $(n-1)$-connected CW complex X with $\pi_n(X) \cong G$. By Lemma 3, we may attach cells of dimension $> n+1$ to obtain a space Y with the same properties and such that $\pi_k(Y) = 0$ for $k > n$, i.e., Y is a $K(G,n)$.

For uniqueness, let Z be a $K(G,n)$. By the construction of Y, we have the diagram

$$\begin{tikzcd}
\bigvee_I S^n \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & Z \\
X \arrow[ur, dashed] \arrow[d, phantom, sloped, "\subseteq"] & \\
Y \arrow[uur, dashed] &
\end{tikzcd}$$

where the second extension uses Lemma 4 and the first is achieved by recalling that X is obtained from $\bigvee_I S^n$ by attaching $(n+1)$-cells along maps that become nullhomotopic in Z by construction.

The induced diagram on π_n is

$$\begin{tikzcd}
\bigoplus_I \mathbb{Z} \arrow[r] \arrow[d] & G \\
G \arrow[ur, dashed, "f_*", "\cong"'] &
\end{tikzcd}$$

so f induces an isomorphism on π_n, hence on π_n for every k, since both spaces are Eilenberg–MacLane spaces. $\square$