PROBLEM LIST 2
TOPOLOGY 3, SPRING 2024

Definition 1. We say that a map $p : E \to B$ has the *lift extension* property for the pair (X, A) if every lifting problem of the following form has a solution:

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow p \\ X & \longrightarrow & B. \end{array}$$

We say that it has the (relative) *homotopy lifting property* for (X, A) if it has the lift extension property for $(X \times [0,1], A \times [0,1] \cup X \times \{0\})$.

Example. The HLP defined in class is the relative HLP for $(X, \varnothing)$. Equivalently, it is the lift extension property for $(X \times [0,1], X \times \{0\})$.

Problem 1. Fix a map $p : E \to B$.

(a) Show that the pairs $(D^n \times [0,1], D^n \times \{0\})$ and $(D^n \times [0,1], S^{n-1} \times [0,1] \cup D^n \times \{0\})$ are homeomorphic. You may use figures to assist you.
(b) Show that p is a Serre fibration if and only if p has the HLP for each of the pairs (D^n, S^{n-1}).
(c) Show that p is a Serre fibration if and only if p has the HLP for all CW pairs (hint: one cell at a time).
(d) Show that p is a Serre fibration if and only if p has the lift extension property for all CW pairs (X, A) with A a deformation retract of X.
(e) Show that p is a Serre fibration if and only if p has the lift extension property for all CW pairs (X, A) such that $A \subseteq X$ is a weak equivalence.

Definition 2. We say that a pair (X, A) has the *homotopy extension* property (HEP) for Y if a homotopy $H : A \times [0,1] \to Y$ can always be extended to a homotopy defined on all of X and agreeing with a prescribed extension at $t = 0$.

Problem 2. Let (X, A) be a pair of spaces.

(a) Construct a map p with domain Y such that (X, A) has the HEP for Y if and only if p has the HLP for (X, A).
(b) Show that a CW pair (X, A) has the HEP for any space Y (hint: don't work too hard).

In the following problem, you may assume that any compact manifold is homotopy equivalent to a CW complex.

Problem 3. Let M be a simply connected, compact, orientable 3-manifold. Show that M is homotopy equivalent to S^3.

Problem 4. Let X be an acyclic CW complex. Show that the suspension of X is contractible.

Problem 5. In this problem, we fix $n > 1$.

(1) Show that every continuous map $f : S^n \to S^1$ is nullhomotopic (hint: think about covering spaces).
(2) Show that every continuous map $f : \mathbb{RP}^n \to S^1$ is nullhomotopic.
(3) Find a continuous map $f : T^2 \to S^1$ that is not nullhomotopic (prove it).

This problem includes Munkres 79.1 and 79.2.

Problem 6. Write $x_0 := (1, 1) \in S^1 \times S^1 = T^2$. Recall that the projections induce a canonical isomorphism $\pi_1(T^2, x_0) \cong \pi_1(S^1, 1) \times \pi_1(S^1, 1) \cong \mathbb{Z} \times \mathbb{Z}$.

(1) Find a covering space of T^2 corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(m, 0)$, where m is a positive integer.
(2) Find a covering space of T^2 corresponding to the trivial subgroup.
(3) Find a covering space of T^2 corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(m, 0)$ and $(0, n)$, where m and n are positive integers.

This problem is Munkres 79.4. In fact, these covering spaces form a complete list—see Munkres 79.5 for more on the subject.