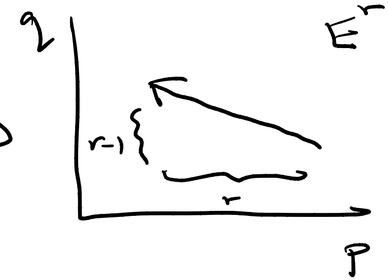
Last time

- Exact and derived couples
- Spectral sequences + bigradings
- Regularity properties of filtrations

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$



Thm (Convergence) Let $C = \bigcup_p C_p$ be ELF and bounded below. Then

$$E^\infty_{p,q} \cong i^\infty(A^1_{p,q}) \big/ i^\infty(A^1_{p-1,q+1}).$$

If C is also complete, then

$$i^\infty(A^1_{p,q}) \cong \mathrm{im}\left(H_{p+q}(C_p) \longrightarrow H_{p+q}(C)\right).$$

Remark Given a filtration of an object S one obtains an **associated graded** object $\operatorname{gr} S$ of the same type by forming the quotients of successive filtration steps.

In our example, we have

$$E^\infty \cong \operatorname{gr} H_*(X),$$

where $H_*(X)$ is filtered by the images of the X_p.

Remark It is natural to set

$$\operatorname{gr} X = \bigvee_p X_p / X_{p-1},$$

in which case $E^1 \cong H_*(\mathrm{gr} X)$. From this point of view, the role of the spectral sequence is to interpolate between $H_*(\mathrm{gr} X)$ and $\mathrm{gr} H_*(X)$.

Example Consider the filtration

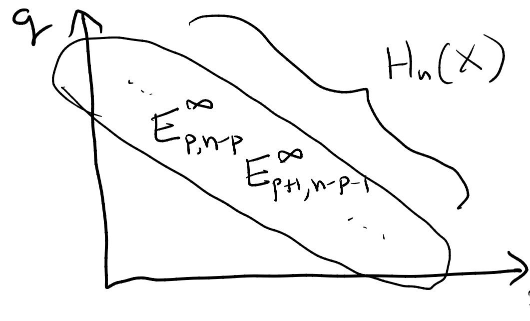
$$\{0\} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}.$$

Then $\mathrm{gr}\,\mathbb{Z} \cong 2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \not\cong \mathbb{Z}$, so a filtered object is not recoverable from the associated graded object.

Exercise If $V = \bigcup V_p$ is a filtered vector space, then $\mathrm{gr} V \cong V$ non-canonically.

__Cor__ Let $X = \bigcup_p X_p$ be a bounded below complete filtration, and fix a field $\mathbb{F}$. There is a non-canonical isomorphism

$$H_n(X;\mathbb{F}) \cong \bigoplus_{p+q=n} E^\infty_{p,q}.$$



__Lemma__ If $C = \bigcup_p C_p$ is bounded below, then $A^r_{p,q} = 0$ for fixed p and r sufficiently large.

__Proof__ Since i sends (p,q) to $(p+1,q-1)$, we have

$$A^r_{pq} = i\left(A^{r-1}_{p-1,q+1}\right) = i^2\left(A^{r-2}_{p-2,q+1}\right) = \cdots = i^{r-1}\left(A^1_{p-r+1,q+r-1}\right),$$

but $A^1_{p-r+1,q+r-1} = H_{p+q}\left(C_{p-r+1}\right) = 0$ if $r \gg 0$.

$\square$

__Proof of theorem__ Consider the exact sequence

$$\cancelto{0}{E^r_{p+r-1,q-r+2}} \longrightarrow A^r_{p+r-2,q-r+2} \longrightarrow A^r_{p+r-1,q-r+1} \longrightarrow E^r_{pq} \to \cancelto{0}{A^r_{p-1,q}} \to \cancelto{0}{A^r_{p,q-1}}.$$

For $r \gg 0$, the leftmost term vanishes by ELF and the two rightmost terms by bounded below, yielding the exact sequence

$$0 \to i^\infty(A^1_{p-1,q+1}) \to i^\infty(A^1_{pq}) \to E^\infty_{p,q} \to 0.$$

Assuming completeness, we have the commuting diagram

$$\begin{tikzcd}
A^1_{p,q} \arrow[r, "i^{r-1}"] \arrow[d, equal] & A^r_{p+r-1,q-r+1} \subseteq A^1_{p+r-1,q-r+1} \arrow[d, equal] & \\
H_{p+q}(C_p) \arrow[r] & H_{p+q}(C_{p+r-1}) \arrow[r, "(*)"] & H_{p+q}(C).
\end{tikzcd}$$

For $r >> 0$, the image of the top map is $i^\infty(A^1_{p,q})$ and $(*)$ is an isomorphism.

$\square$

A spectral sequence is typically only useful if E^2 can be identified with something meaningful, which depends on the situation at hand. In the case of a fibration, we require one further observation.

Fix a map $\pi: E \to B$, and write $F_b = \pi^{-1}(b)$.

<u>Lemma</u> If π is a Hurewicz (resp. Serre) fibration and b_1 and b_2 lie in the same path component of B, then F_{b_1} and F_{b_2} are (weakly) homotopy equivalent.

Thus, it is sensible to speak of <u>the</u> fiber.

<u>Thm</u> (Serre 1.0) Let $\pi: E \to B$ be a fibration with B a simply connected CW complex. There is a spectral sequence

$$E^2_{p,q} \cong H_p(B; H_q(F)) \Longrightarrow H_{p+q}(E),$$

where $F = F_b$ for some $b \in B$.

<u>Rmk</u> The assumption that B is a CW complex is not essential.

<u>Rmk</u> We will eventually remove the

assumption that B be simply connected.

<u>Ex</u> Consider the Hopf bundle

$$S^1 \to S^3 \to S^2$$

The Serre spectral sequence has

$$E^2_{p,q} \cong H_p(S^2; H_q(S^1)) \cong \begin{cases} \mathbb{Z} & p=0,2,\ q=0,1 \\ 0 & \text{otherwise} \end{cases}$$

E^2

$$\begin{array}{c|ccccc}
q & & & & & \vdots \\
2 & 0 & 0 & 0 & 0 & \cdots \\
1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\
0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\
\hline
 & 0 & 1 & 2 & & p
\end{array}$$

Of course, we know $H_*(S^3)$, but suppose we knew only that S^3 is simply connected, which is much easier.

<u>Claim</u> $d^2_{2,0}$ is an isomorphism.

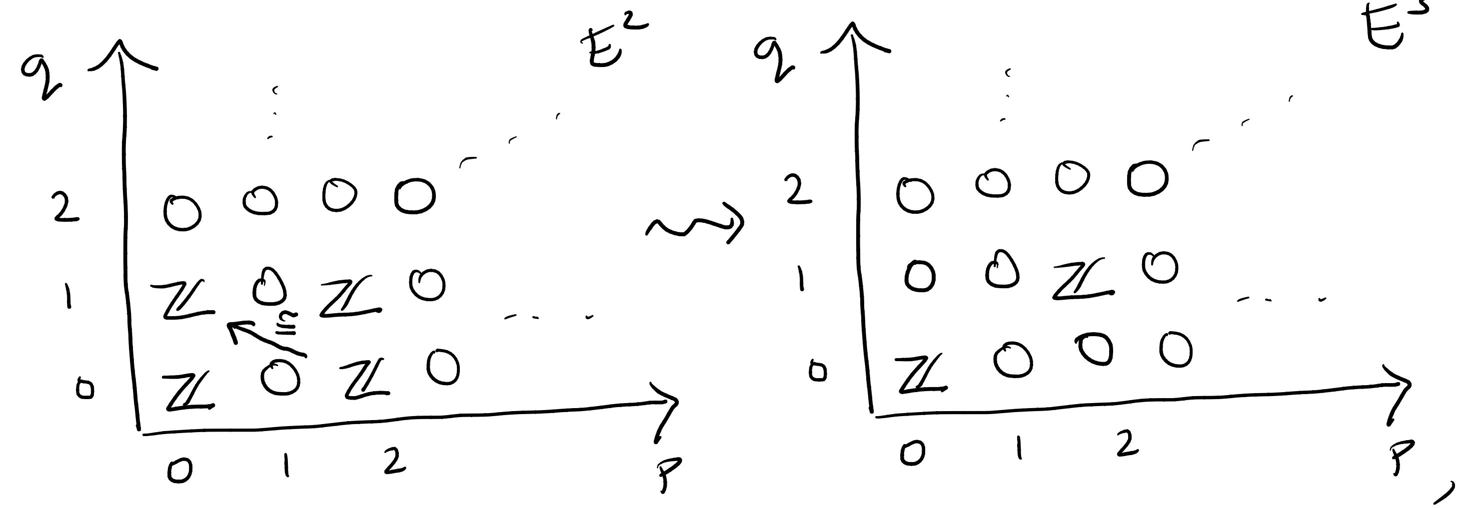
<u>Proof</u> For degree reasons, $d^2_{p,q} = 0$ for $(p,q) \neq (2,0)$, and $d^r \equiv 0$ for $r > 2$.

If $d^2_{2,0}$ is not surjective, then

$$0 \neq E^3_{0,1} = E^4_{0,1} = \cdots = E^\infty_{0,1} \Rightarrow H_1(S^3) \neq 0.$$

$\square$

Thus, the spectral sequence "collapses" at $E^3 = E^\infty$:



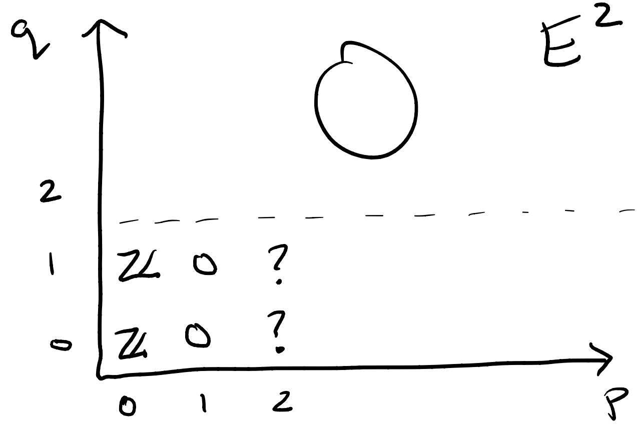
and we read off $H_*(S^3)$.

This easy example is a "forward" application, turning information about B and F into information on E. "Backward" applications are also common.

<u>Ex</u> Consider the bundle

$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{CP}^\infty,$$

and assume $H_*(\mathbb{CP}^\infty)$ is unknown. Then $E^2_{p,q} = H_p(\mathbb{CP}^\infty; H_q(S^1))$ is only partially known; since $\mathbb{CP}^\infty$ is simply connected, we have



Since S^∞ is contractible, $E^\infty_{p,q} = 0$ for $(p,q) \neq (0,0)$, so the argument from before shows that $d^2_{2,0}$ is surjective.

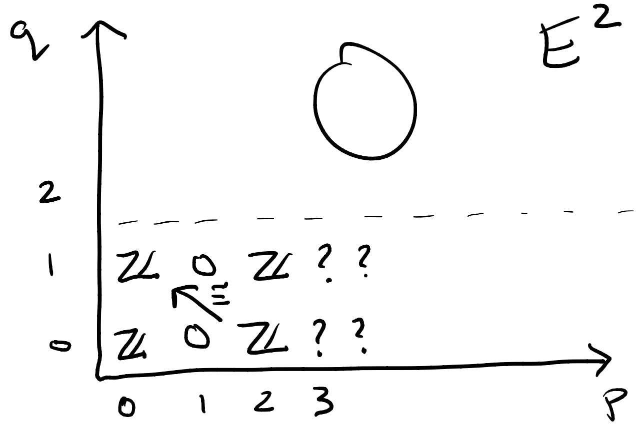
<u>Claim</u> $d^2_{2,0}$ is injective

<u>Proof</u> The only nonzero differential with source or target $E^r_{2,0}$ is $d^2_{2,0}$, so

$$\ker d^2_{2,0} = E^3_{2,0} = E^4_{2,0} = \cdots = E^\infty_{2,0} = 0$$

$\square$

Thus, $H_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$:



__Claim__ $d^2_{2n,0}$ is an isomorphism

__Proof__ By induction, $H_{2n-2}(\mathbb{CP}^\infty) \cong \mathbb{Z}$, so failure of surjectivity implies

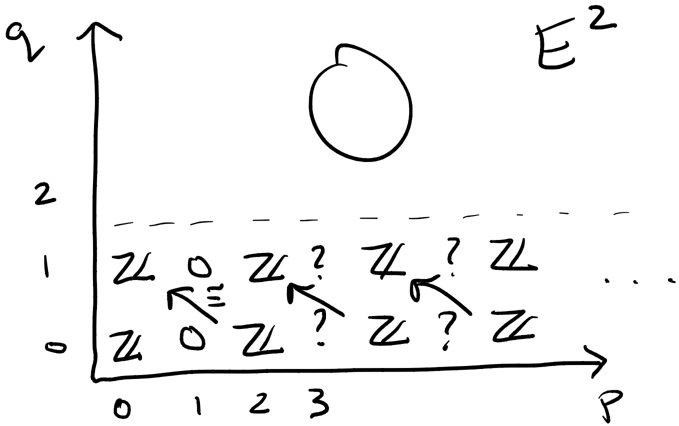
$$0 \neq E^3_{2n-2,2} = \cdots = E^\infty_{2n-2,2} = 0,$$

and failure of injectivity implies

$$0 \neq E^3_{2n,0} = E^3_{2n,0} = \cdots = E^\infty_{2n,0} = 0 .$$

□

Thus, $H_{2n}(\mathbb{CP}^\infty) \cong \mathbb{Z}$ for $n \geq 0$.



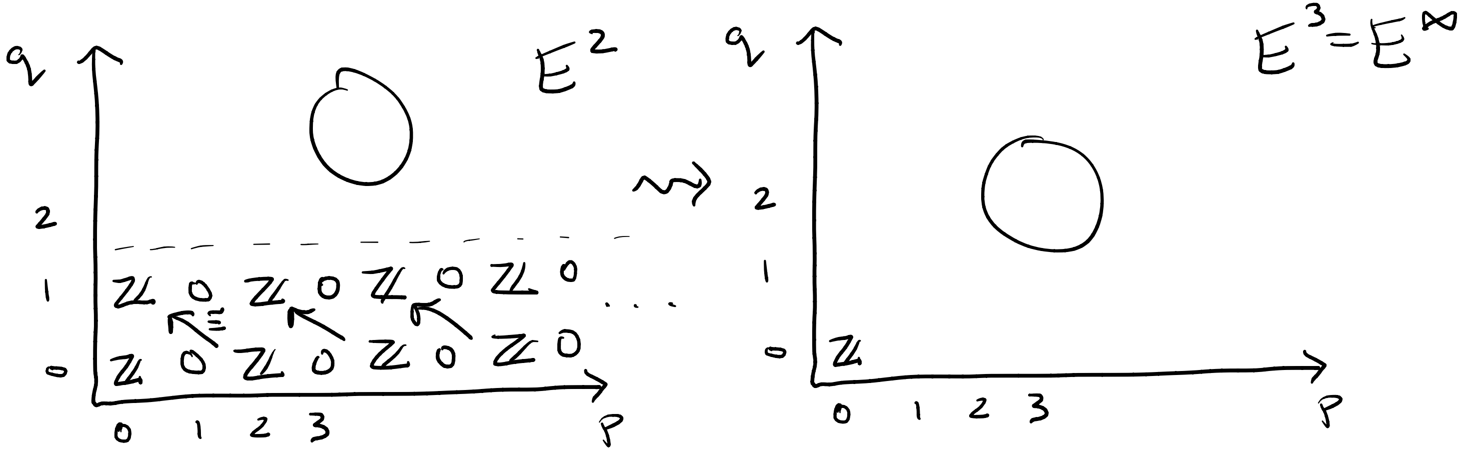
<u>Claim</u> $H_{2n+1}(\mathbb{CP}^\infty) = 0$ for $n \geq 0$.

<u>Proof</u> By induction, $H_{2n-1}(\mathbb{CP}^\infty) = 0$,

so no nonzero differential has source or target $E^r_{2n+1,0}$, whence

$$E^2_{2n+1,0} = \cdots = E^\infty_{2n+1,0} = 0.$$

$\square$



This technique gives new calculations that would otherwise be impossible.