Last time

- Associated graded
- Convergence theorem
- Serre spectral sequence
- Toy examples

$$H_*(\mathrm{gr}\, C) \Rightarrow \mathrm{gr}\, H_*(C)$$

Thm For $n > 0$, we have

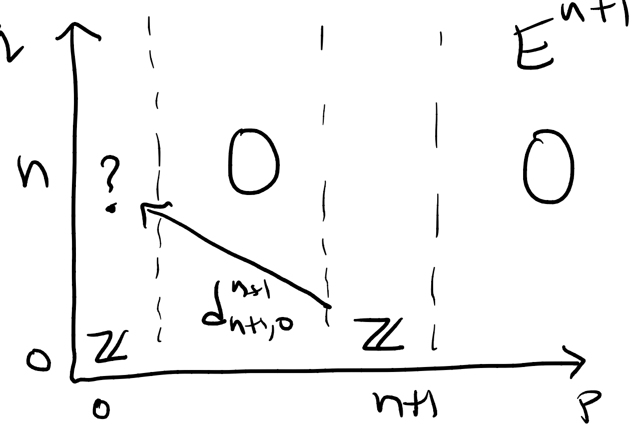
$$H_i(\Omega S^{n+1}) \cong \begin{cases} \mathbb{Z} & n \mid i \\ 0 & \text{otherwise}. \end{cases}$$

<u>Rmk</u> $\Omega S^1 \simeq \mathbb{Z}$ (a countable discrete space).

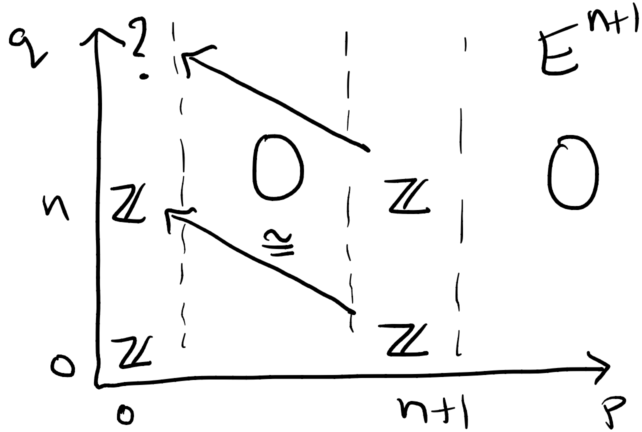
<u>Proof</u> Consider the path fibration

$$\Omega S^{n+1} \to PS^{n+1} \to S^{n+1}.$$

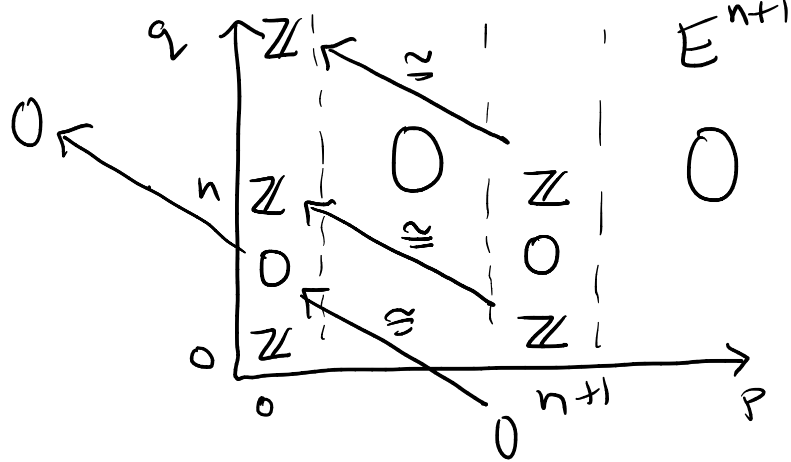
Since PS^{n+1} is contractible, $E^{\infty}_{p,q} = 0$ for $(p,q) \neq (0,0)$. For degree reasons, the only possible nonzero differentials are those of the form $d^{n+1}_{n+1,q}$



As before, a contradiction at E^∞ results if $d^{n+1}_{n+1,0}$ is not an isomorphism.



The same inductive argument shows $d^{n+1}_{n+1,rn}$ is an isomorphism for every $r \geq 0$, so $H_{rn}(\Omega S^{n+1}) \cong \mathbb{Z}$. For the rest, no nonzero differential enters or exits the $p=0$ column for $q<n$, so these entries must be zero to avoid contradiction:



The same induction as before shows that all remaining entries in the $p=0$ column must vanish, as desired.

$\square$

The Serre spectral sequence also gives partial information on more general loop spaces.

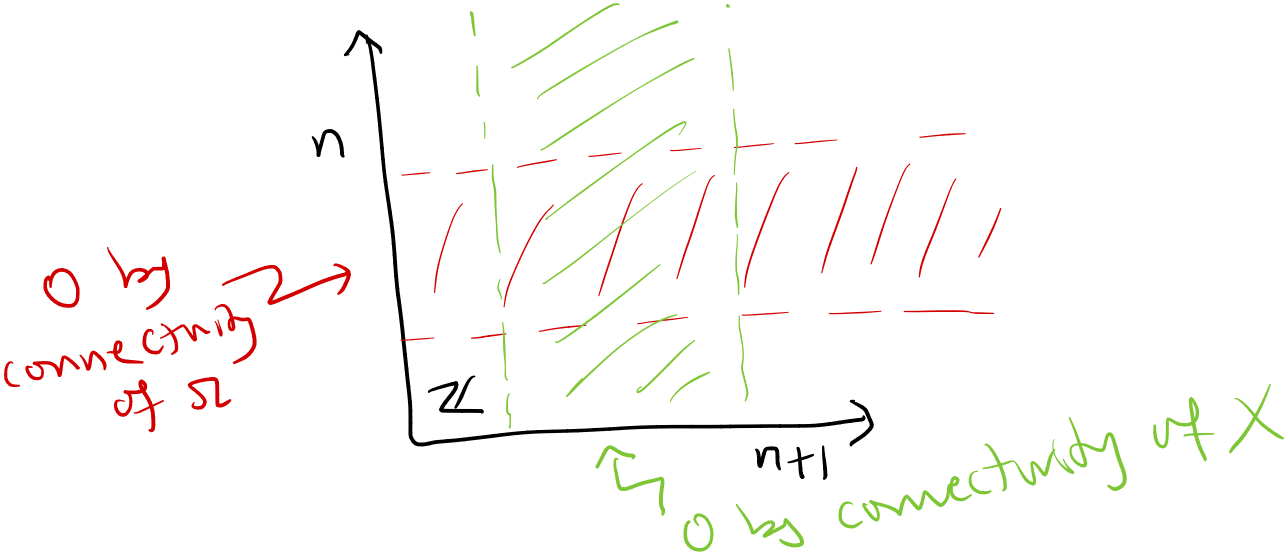
<u>Thm</u> If X is n-connected, then

$$d^m_{m,0} : H_m(X) \xrightarrow{\cong} H_{m-1}(\Omega X)$$

for $m \leq 2n$.

Proof The only potential differentials with target in bidegree $(0, m-1)$ are of the form $d^r_{r,m-r}$ for $1 < r \leq m$, so it suffices to show that $E^2_{r,m-r} = 0$ for $r < m \leq 2n$.

Since $E^2_{r,m-r} \cong H_r(X; H_{m-r}(\Omega X))$, and since X is n-connected, it suffices to note that $H_{m-r}(\Omega X) = 0$ for $n \leq r < m \leq 2n$, i.e., for $m-r < n$, since ΩX is n-connected. $\square$



Cor (Freudenthal) If X is $(n-1)$-connected, the suspension map $\pi_{m-1}(X) \to \pi_m(\Sigma X)$ is an isomorphism for $m < 2n$.

Proof By currying, $[\Sigma X, Y] \cong [X, \Omega Y]$. Taking $Y = \Sigma X$, we obtain a map $\eta : X \to \Omega \Sigma X$, and it is not difficult, though somewhat laborious to show that the following diagram commutes:

$$
\begin{tikzcd}
H_m(\Sigma X) \arrow[r, "d^m_{m,0}"] \arrow[d, "\mathrm{susp}"', "\cong"] & H_{m-1}(\Omega\Sigma X) \\
H_{m-1}(X) \arrow[ur, "\eta_*"] & \pi_{m-1}(\Omega\Sigma X) \arrow[u, "h"'] \arrow[d, "\cong"] \\
\pi_{m-1}(X) \arrow[u, "h"] \arrow[ur, "\eta_*"] \arrow[r, "\mathrm{susp}"] & \pi_m(\Sigma X)
\end{tikzcd}
$$

Applying relative Hurewicz completes the proof. $\square$

We move toward justifying the SSS.

<u>Def</u> The <u>pullback</u> of $\pi: E \to B$ along $f: A \to B$ is

$$f^*E := A \times_B E = \{(a,e) \in A \times E \mid f(a) = \pi(e)\}.$$

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

<u>Ex</u> If $i_b : \{b\} \hookrightarrow B$ is the inclusion, then $i_b^* E \cong F_b := \pi^{-1}(b)$.

Lemma The pullback of a fibration along any map is a fibration.

Proof Exercise. $\square$

Def Given $\pi_i : E_i \to B$, a map $f : E_1 \to E_2$ is <u>fiber-preserving</u> if $\pi_2 \circ f = \pi_1$.

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[dr, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ & B \end{tikzcd}$$

It is a <u>fiber homotopy equivalence</u> if there is a second fiber-preserving map $g : E_2 \to E_1$ and fiber-preserving homotopies $fg \simeq id_{E_2}$ and $gf \simeq id_{E_1}$.

For the remainder of the lecture, "fibration" will mean "Hurewicz fibration".
The same results hold for Serre fibrations with homotopy equivalence replaced by weak homotopy equivalence.

Prop Let $\pi: E \to B$ be a fibration and $H: A \times [0,1] \to B$ a homotopy. The pullbacks $H_0^* E$ and $H_1^* E$ are fiber homotopy equivalent.

Proof Given $\gamma : [0,1] \to [0,1]$, find the lift

$$\begin{tikzcd}
H^{\star}_{\gamma(0)}E \arrow[r, hook] \arrow[d, hook, "0"'] & H^{\star}E \arrow[d] \\
H^{\star}_{\gamma(0)}E \times [0,1] \arrow[r, "\pi \times \gamma"] \arrow[ur, dashed, "F"] & A \times [0,1]
\end{tikzcd}$$

Then $\operatorname{im} F_1 \subseteq H^{\star}_{\gamma(1)}E$, and we obtain a fiber-preserving map $L_\gamma : H^{\star}_{\gamma(0)}E \to H^{\star}_{\gamma(1)}E$. Using HLP again (exercise), we have that $\gamma \simeq_p \gamma' \Rightarrow L_\gamma \simeq_{FP} L_{\gamma'}$ and $L_{\gamma_1 \star \gamma_2} \simeq_{FP} L_{\gamma_1} \circ L_{\gamma_2}$. Taking $\gamma_1 = id_{[0,1]}$ and $\gamma_2 = \overline{id}_{[0,1]}$ gives the result. $\square$

Cor If B is contractible and $\pi : E \to B$ is a fibration, then E is fiber homotopy equivalent to $B \times F_b$ for any $b \in B$.

Proof If $H : B \times [0,1] \to B$ be a homotopy from id_B to the constant map at b, then $H_0^* E \cong E$ and $H_1^* E \cong B \times F_b$. $\square$

Cor A fibration over a locally contractible base is fiber homotopically locally trivial.

Cor If $\pi : E \to B$ is a fibration, then $F_{b_1} \simeq F_{b_2}$ if b_1 and b_2 lie in the same path component.

<u>Proof</u> A path $\gamma : [0,1] \to B$ is a homotopy from i_{b_1} to i_{b_2}, so

$$F_{b_1} \cong i_{b_1}^* E \simeq i_{b_2}^* E \cong F_{b_2}.$$

$\square$

More specifically, we have

$$\left\{ \begin{matrix} \text{paths from} \\ b_1 \text{ to } b_2 \end{matrix} \right\} \Big/ \simeq_p \longrightarrow \left\{ \begin{matrix} \text{homotopy} \\ \text{equivalences} \\ F_{b_1} \simeq F_{b_2} \end{matrix} \right\} \Big/ \simeq,$$

$$[\gamma] \longmapsto [L_\gamma]$$

and, taking $b_1 = b_2 = b$, we have a group homomorphism

$$\begin{array}{ccccc}
\left\{\begin{matrix}\text{loops at}\\ b\end{matrix}\right\}\Big/\simeq_p & \longrightarrow & \left\{\begin{matrix}\text{self-homotopy}\\ \text{equivalences of } F_b\end{matrix}\right\}\Big/\simeq & \longrightarrow & \left\{\begin{matrix}\text{automorphisms}\\ \text{of } H_*(F_b)\end{matrix}\right\}\\
\shortparallel & & \shortparallel & & \shortparallel\\
\pi_1(B,b) & \longrightarrow & \mathrm{Aut}^h(F_b) & \longrightarrow & \mathrm{Aut}(H_*(F_b))
\end{array}$$

<u>Upshot</u> If $\pi: E \to B$ is a fibration, there is a canonical action of $\pi_1(B,b)$ on $H_*(F_b)$.

<u>Ex</u> If π is a covering map, we recover the action of $\pi_1(B,b)$ on $\pi^{-1}(b)$ by path lifting.

In particular, this action can be non-trivial.

<u>Thm</u> (Serre 2.0) Let $\pi : E \to B$ be a fibration with B a connected CW complex. If $\pi_1(B)$ acts trivially on $H_*(F)$, where $F = F_b$ for some $b \in B$, then there is a spectral sequence

$$E^2_{p,q} \cong H_p(B; H_q(F)) \Longrightarrow H_{p+q}(E).$$

<u>Rmk</u> Nontrivial actions can be accommodated by homology with local coefficients.

<u>Rmk</u> We recover the simply connected case.