Last time

- $H_*(\Omega S^{n+1})$
- More on fibrations
- $H_*(F) \circlearrowleft \pi_1(B)$

Serre 2.0 If $\pi_1(B)$ acts trivially on $H_*(F)$, then there is a spectral sequence

$$H_p(B; H_q(F)) \Longrightarrow H_{p+q}(E).$$

The spectral sequence is the one arising from the filtration $E = \bigcup_{p \geq 0} E_p$, where $E_p = \pi^{-1}(B_p)$. In particular, this filtration is bounded below.

<u>Lemma 1</u> The filtration $E = \bigcup_{p \geq 0} E_p$ is complete.

Thus, E^∞ is defined and isomorphic to $\operatorname{gr} H_*(E)$.

It remains to identify E^2.

<u>Lemma 2</u> For $p > 0$, (E_p, E_{p-1}) is good.

<u>Proof</u> Exercise (HLP and (B_p, B_{p-1}) is so). $\square$

Lemma 3 $E^1_{p,q} \cong C^{cw}_p(B; H_q(F))$.

Proof of Lemma 1 Since $H_n(E_p, E_{p-1}) = 0$ for $p > n$ by Lemma 3, it suffices to note that a compact subspace $K \subseteq E$ lies in some E_p, since $\pi(K)$ intersects finitely many cells of B. $\square$

Proof of Lemma 3 We have

$$E^1_{p,q} := H_{p+q}(E_p, E_{p-1}) \cong \tilde{H}_{p+q}(E_p/E_{p-1}).$$

Let $e_\alpha : D^p \longrightarrow B$ be characteristic maps for the p-cells of B. Since D^p is contractible, $e_\alpha^* E$ is fiber homotopy equivalent to $D^p \times F_\alpha$, where F_α is the fiber over $e_\alpha(0)$.

$$E_p / E_{p-1} \cong \frac{\coprod_\alpha e_\alpha^* E}{\coprod_\alpha (e_\alpha|_{S^{p-1}})^* E} \overset{!}{\simeq} \frac{\coprod_\alpha D^p \times F_\alpha}{\coprod_\alpha S^{p-1} \times F_\alpha}$$

$$\simeq \frac{\coprod_\alpha D^p \times F}{\coprod_\alpha S^{p-1} \times F},$$

where the last is induced by choosing paths γ_α from b to $e_\alpha(0)$ (B is connected).

We have the long exact sequence

$$\begin{tikzcd}
H_{p+q}(S^{p-1}\times F) \arrow[r] & H_{p+q}(D^p\times F) \arrow[l, bend left] \arrow[r, "0"] & \tilde{H}_{p+q}(D^p\times F/S^{p-1}\times F) \arrow[d] \\
& & H_{p+q-1}(S^{p-1}\times F) \arrow[d, "i_*"'] \\
& & H_{p+q-1}(D^p\times F) \arrow[u, bend right]
\end{tikzcd}$$

A choice of point in S^{p-1} gives the indicated sections, so

$$\tilde{H}_{p+q}(D^p\times F/S^{p-1}\times F) \cong \ker(i_*).$$

By the Künneth theorem, we have the commuting diagram

$$
\begin{tikzcd}
H_{p+q-1}(S^{p-1}\times F) \arrow[r, "i_*"] \arrow[d, equal, "\wr"'] & H_{p+q-1}(D^p\times F) \arrow[d, equal, "\wr"] \\
\begin{array}{c} H_0(S^{p-1})\otimes H_{p+q-1}(F) \\ \oplus \\ H_{p-1}(S^{p-1})\otimes H_q(F) \end{array} \arrow[r, "\cong \oplus 0"] & \begin{array}{c} H_0(D^p)\otimes H_{p+q-1}(F) \\ \oplus \\ \cancelto{0}{H_{p-1}(D^p)\otimes H_q(F)} \end{array}
\end{tikzcd}
$$

So $\tilde{H}_{p+q}(D^p\times F/S^{p-1}\times F) \cong H_q(F)$, whence

$$E^1_{p,q} \cong \bigoplus_\alpha \tilde{H}_{p+q}(D^p\times F/S^{p-1}\times F) \cong \bigoplus_\alpha H_q(F),$$

as desired. $\square$

The proof will be complete after proving the following result.

<u>Lemma 4</u> The following diagram commutes:

$$\begin{tikzcd} E^1_{p,q} \arrow[r, equal, "\sim"] \arrow[d, "d^1"'] & C^{CW}_p(B; H_q(F)) \arrow[d, "d^{CW}"] \\ E^1_{p-1,q} \arrow[r, equal, "\sim"] & C^{CW}_{p-1}(B; H_q(F)) \end{tikzcd}$$

<u>Def</u> Let X be a CW complex and $e_\alpha : D^p \to X$ a p-cell. We say that e_α is <u>standard</u> relative to the $(p-1)$-cell $e_\beta : D^{p-1} \to X$ if there exist closed subsets $D_1, \dots, D_k \subseteq S^{p-1}$ such that

(1) e_α restricts to a homeomorphism $\mathring{D}_i \cong e_\beta(\mathring{D}^{p-1})$ for $1 \le i \le k$

(2) $e_\alpha\left(S^{p-1} \setminus \bigcup_{i=1}^{k} \mathring{D}_i\right) \subseteq X_{p-1} \setminus e_\beta(\mathring{D}^{p-1})$.

<u>Lemma 5</u> Fixing α and β, X is homotopy equivalent to a CW complex with the same cells such that e_α is standard relative to e_β.

Proof If $\partial e_\alpha^{-1}(e_\beta(\mathring{D}^{p-1})) = \emptyset$, there is nothing to show, so we may assume there is a closed disk in S^{p-1} mapped into $e_\beta(\mathring{D}^{p-1})$. By smooth approximation and HEP, we may assume that e_α is smooth on the interior of this disk after replacing X by a homotopy equivalent CW complex with the same cells. By Sard's theorem and the inverse function theorem, there are closed subsets $D_1, \dots, D_n$ such that

$e_\alpha|_{\mathring{D}_i}$ is a homeomorphism onto a fixed open ball. By radial expansion, we may take this ball to be all of $e_\beta(\mathring{D}^{p-1})$. By HEP, this new attaching map yields a homotopy equivalent CW complex with the same cells. For details, compare Bredon IV.11.2. $\square$

<u>Lemma 6</u> Let $\pi_i : E_i \to D^p$ be fibrations for $i \in \{1,2\}$ fitting into the commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ D^p \arrow[r, "g"] & D^p \end{tikzcd}$$

If g is a homeomorphism and f a fiber homotopy equivalence, then the following commutes:

$$\begin{tikzcd}
H_{p+q}(E_1, \partial E_1) \arrow[r, "f_*"] \arrow[d, "\cong"'] & H_{p+q}(E_2, \partial E_2) \arrow[d, "\cong"] \\
H_{p+q}(D^p \times F_1, S^{p-1} \times F_1) \arrow[r, "(g\times f)_*"] \arrow[d, "\cong"'] & H_{p+q}(D^p \times F_2, S^{p-1} \times F_2) \arrow[d, "\cong"] \\
H_q(F_1) \arrow[r, "\deg(\bar{g})\cdot f_*"] & H_q(F_2)
\end{tikzcd}$$

where $\partial E_i := \pi_i^{-1}(S^{p-1})$, $F_1 = \pi_1^{-1}(0)$, $F_2 = \pi_1^{-1}(g(0))$, and $\bar{g}: S^p \to S^p$ is the map induced by g.

<u>Proof</u> By HLP, we may take g to be the identity or a reflection according to whether $\deg(\bar{g})$ is 1 or -1. A long exact sequence reduces to the case $p=1$, which is an exercise. $\square$